Dimension reduction of high-dimensional dynamics on networks with adaptation

Vincent Thibeault, Marina Vegué, Antoine Allard, and Patrick Desrosiers

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Département de physique, de génie physique, et d'optique
Université Laval, Québec, Canada

NETWORKS 2021

Dynamica

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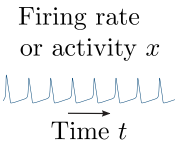
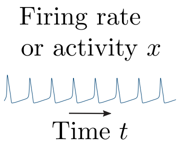
Emergence of collective phenomena (synchronization)

https://www.youtube.com/watch?v=tRPuVAVXk2M


Firing rate
or activity x
Time t



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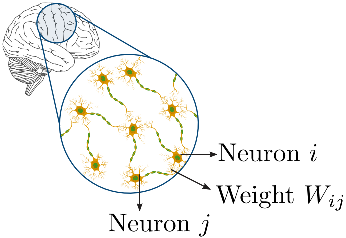
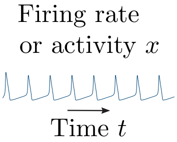


Neuron i
Weight W_{ij}
Neuron j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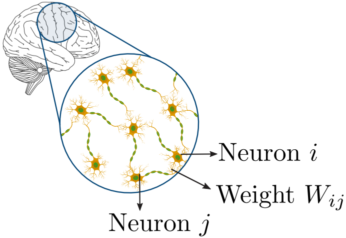
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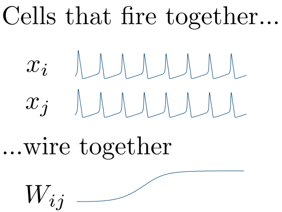
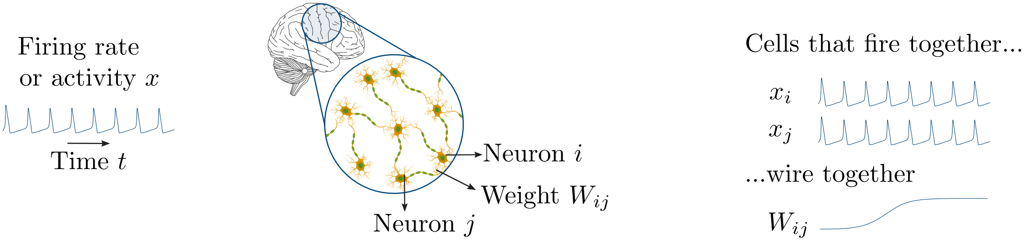

Cells that fire together...
x_i
x_j
...wire together
W_{ij}



Nonlinear activity dynamics + **Complex network** + **Nonlinear adaptation (plasticity)**

$$\frac{dx_i}{dt} = F(x_i) + G(x_i, \textstyle\sum_{j=1}^{N} W_{ij} x_j)$$

$$\frac{dW_{ij}}{dt} = H(x_i, x_j, W_{ij})$$

Complete dynamics

$$N(N+1) \gg 1$$

$$\dot{x}_i = F(x_i) + G(x_i, \sum_{j=1}^{N} W_{ij} x_j)$$
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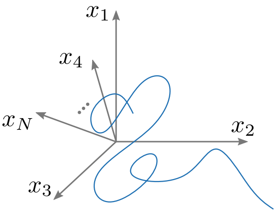

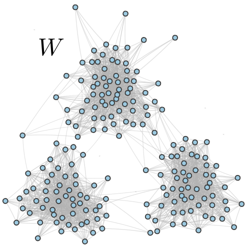

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$$\dot{\mathcal{X}}_\mu \approx ?$$

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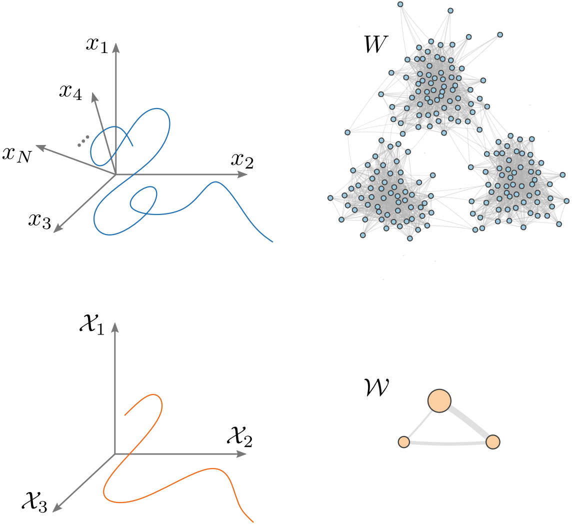

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Dimension reduction allows to ...

- find meaningful global variables $\mathcal{X}_\mu, \mathcal{W}_{\mu\nu}$ (e.g., synchro, global activity,...)

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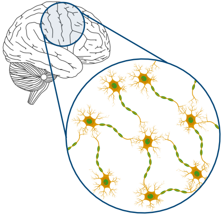

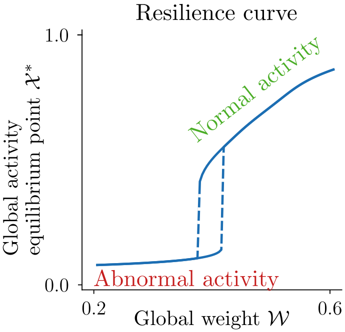

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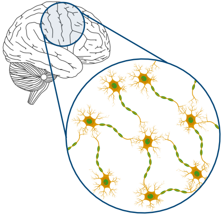

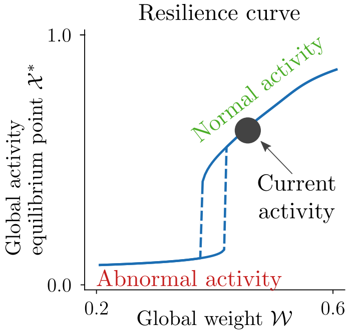

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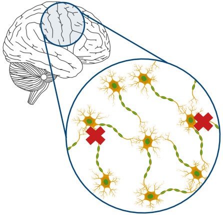

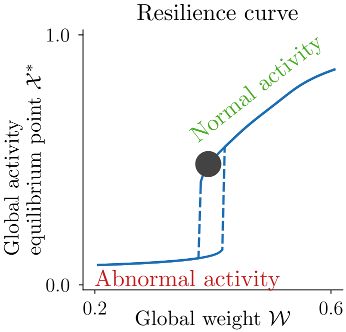

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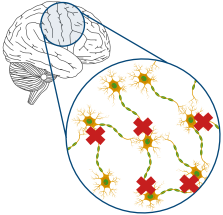

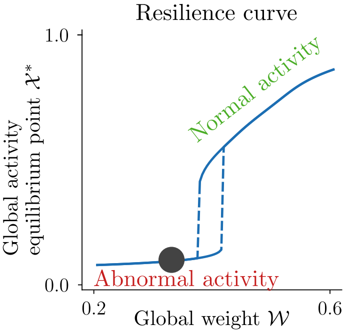

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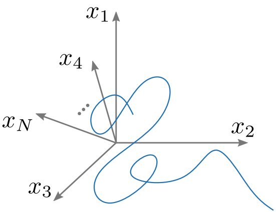

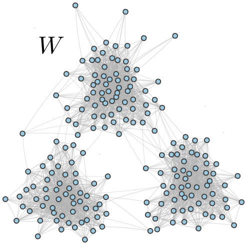

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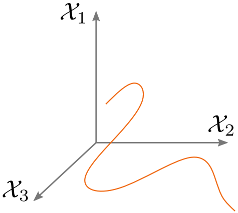

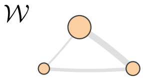

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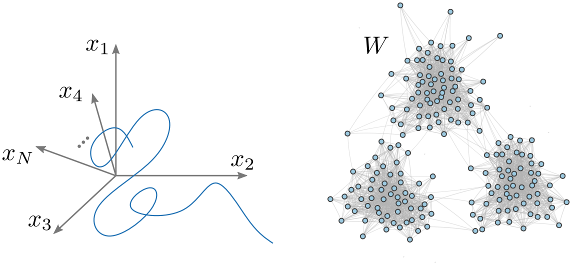

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$$\dot{\mathcal{X}}_\mu \approx F(\mathcal{X}_\mu) + G(\mathcal{X}_\mu, \textstyle\sum_{\nu=1}^{n} \mathcal{W}_{\mu\nu} \mathcal{X}_\nu)$$
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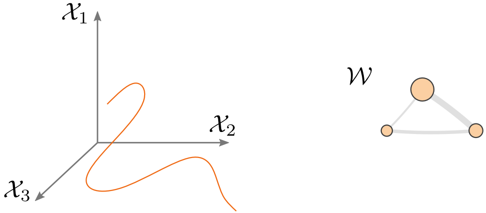

We found $n + n^2$ **linear observables (functions, measures,...)**

$$\mathcal{X}_\mu = \sum_{i=1}^{N} M_{\mu i} x_i,$$

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that both depend on only *one* $n \times N$ matrix.

M is a *reduction matrix* **to be determined**.

Hypothesis

Important neurons contribute strongly to the global activity

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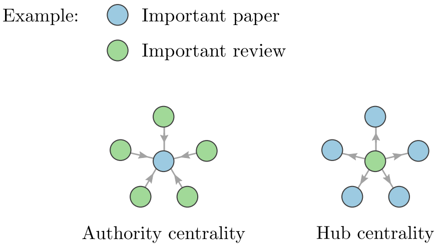

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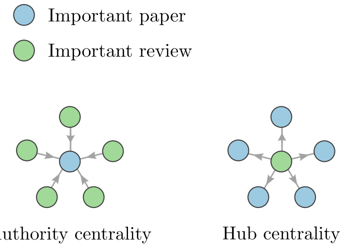
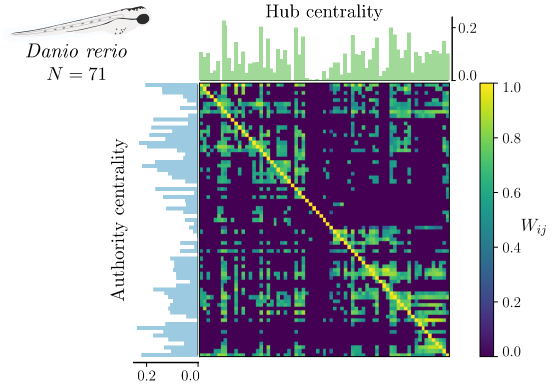
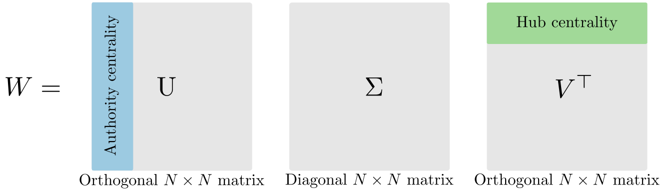
Example:





Singular value decomposition (SVD)



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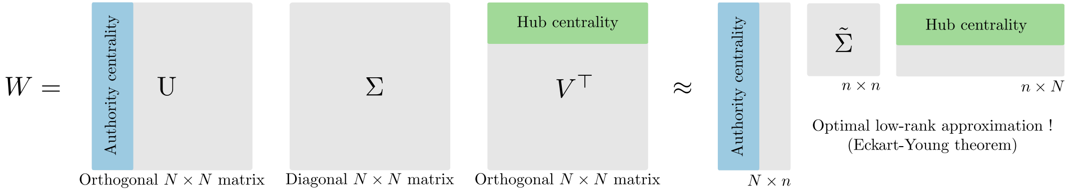

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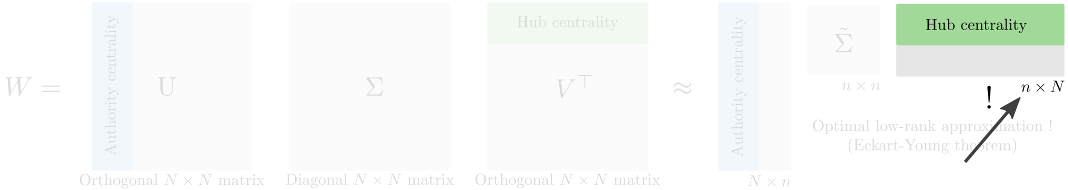

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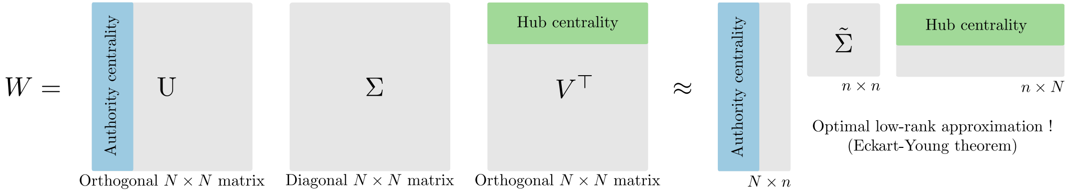
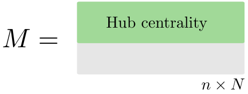


Reduction matrix



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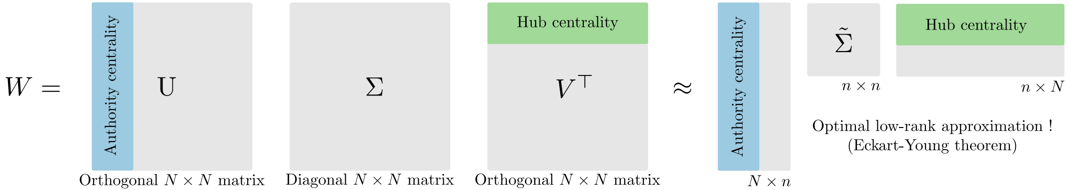

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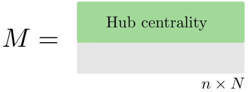

Linear observables

$$\boldsymbol{\mathcal{X}} = M\mathbf{x}$$

$$\mathcal{W} = MWM^\top$$

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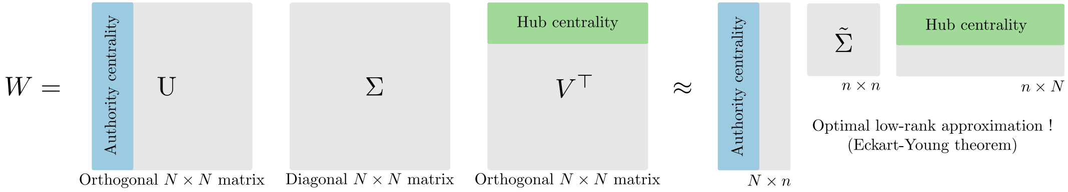

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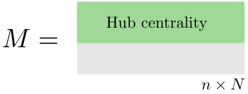

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1. Get equilibrium points for all μ, ν : $\mathcal{X}_\mu^*, \mathcal{W}_{\mu\nu}^*$

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1. Get equilibrium points for all μ, ν : $\mathcal{X}_\mu^*$, $\mathcal{W}_{\mu\nu}^*$
2. Combine these equilibrium points to get the global activities and weights :

$$\mathcal{X}^* = a_1\mathcal{X}_1^* + ... + a_n\mathcal{X}_n^*$$
$$\mathcal{W}^* = b_{11}\mathcal{W}_{11}^* + b_{12}\mathcal{W}_{12}^* + ... + b_{nn}\mathcal{W}_{nn}^*$$

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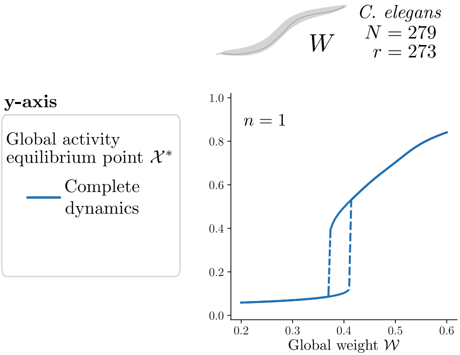
3. Plot resilience curves $\mathcal{X}^*$ vs. $\mathcal{W}^*$

Activity dynamics on a real network without plasticity

C. elegans
$N = 279$
$r = 273$

Activity dynamics on a real network without plasticity



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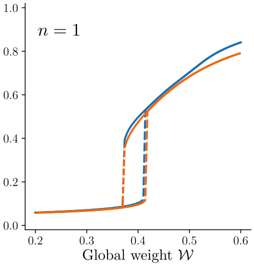
y-axis

Global activity equilibrium point $\mathcal{X}^*$

Complete dynamics

Reduced dynamics



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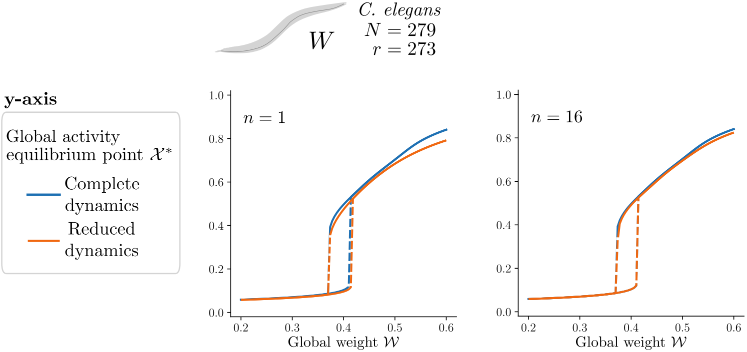

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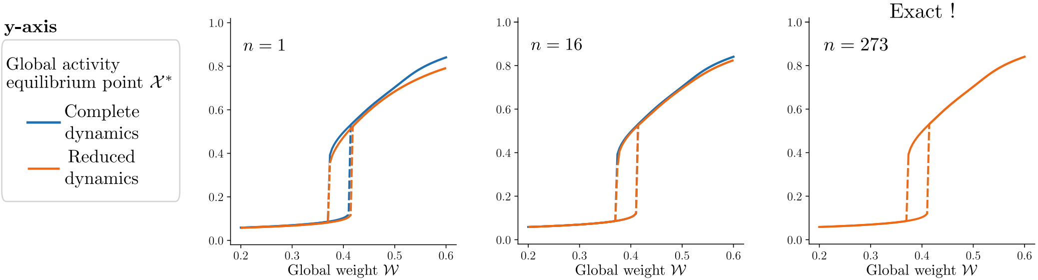
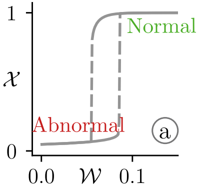


Activity dynamics on an Erdős-Rényi network with plasticity

Complete dynamics : 10 200 ODEs

Reduced dynamics : 3 ODEs



No plasticity

Complete dynamics : 10 200 ODEs

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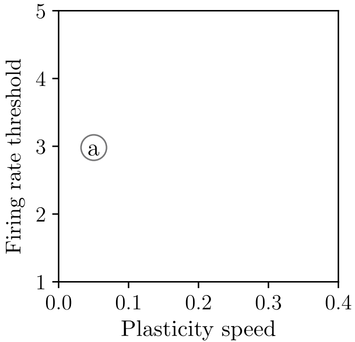

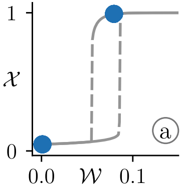

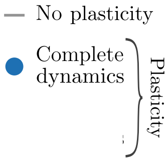

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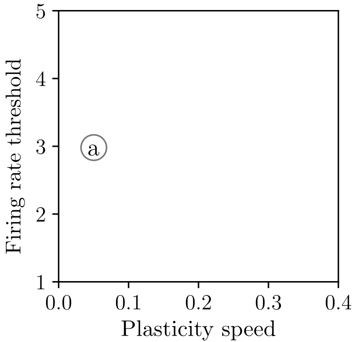

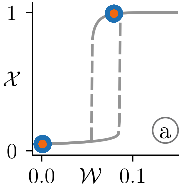

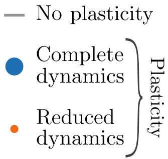

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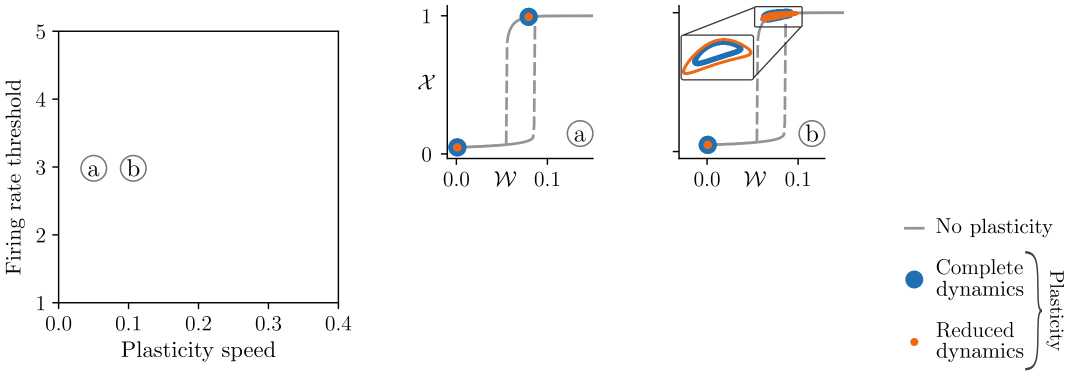

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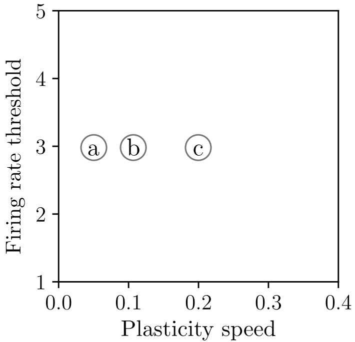

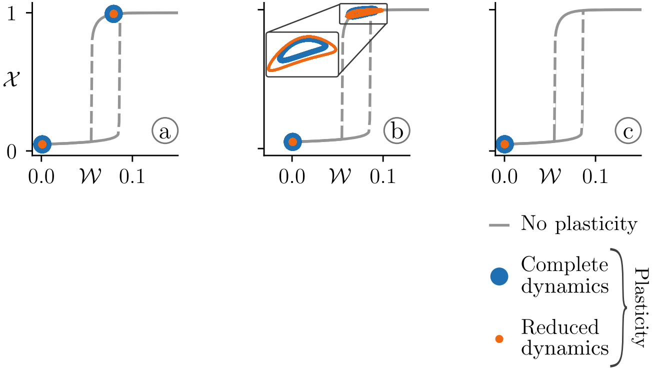

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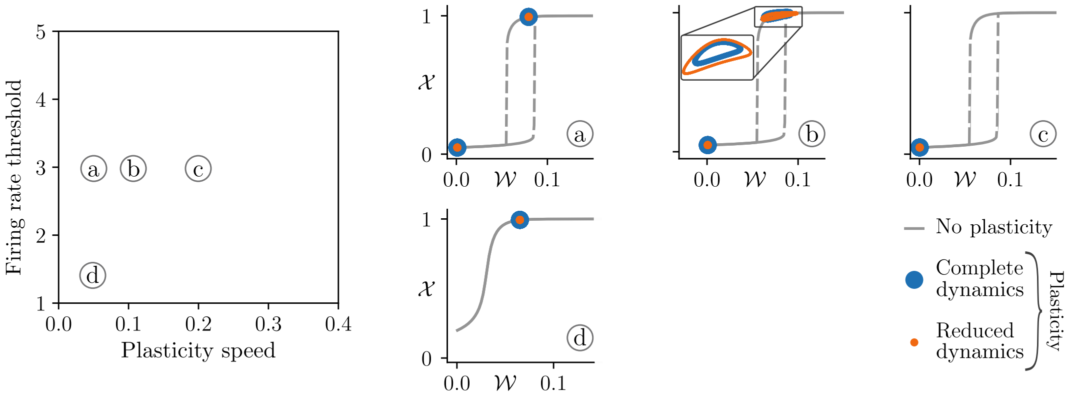

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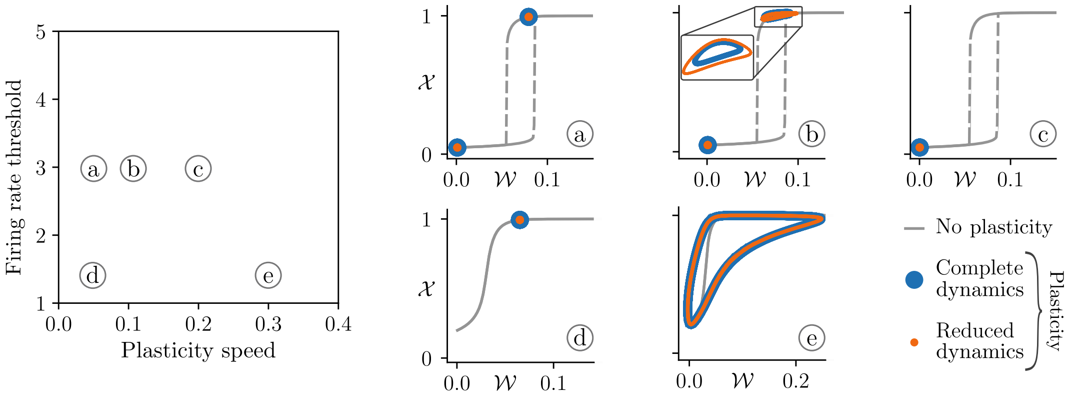

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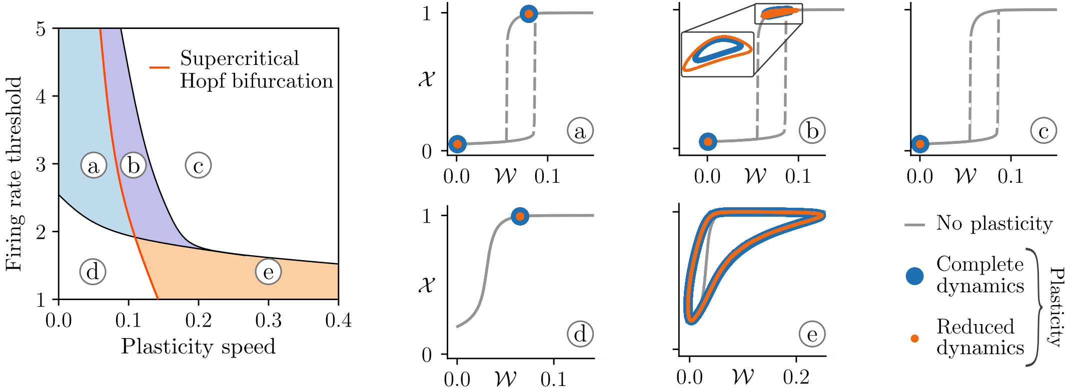

Next steps

- Treat plasticity + real networks ;
- Consider inhibitors ($W_{ij} < 0$) ;
- Use nonlinear observables ;
- Get more profound insights on resilience.

Take home messages

- Reduced dynamics are valuable to disentangle dynamics with plasticity ;
- SVD is a powerful and *interpretable* tool for dimension reduction *of dynamics*.

References and acknowledgments

Thank you for your attention !

Thanks to the organizers !

Questions ?

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Coauthors : M.Vegué, A. Allard, P. Desrosiers

Contact : vincent.thibeault.1@ulaval.ca

Website : `https://dynamicalab.github.io/`

CIMMUL Centre Interdisciplinaire en Modélisation Mathématique de l'Université Laval

Dynamica

CRSNG NSERC

Fonds de recherche Nature et technologies Québec

Université Laval

Centre de recherche CERVO Brain Research Centre Québec

Sentinelle Nord

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In this model, F is linear and G is a sigmoid function :

$$\tau_x \dot{x}_i = -x_i + 1/(1 + e^{-a(y_i - b)}), \quad \text{with} \quad y_i = \sum_{j=1}^{N} W_{ij} x_j$$

- x_i : Firing rate of neuron or brain region i
- τ_x : Time scale of the firing rate
- a : Steepness of the activation function
- b : Firing rate threshold

This model is more complex :

$$\tau_x \, \dot{x}_i = -\alpha_i x_i + \beta_i/(1 + e^{-a(y_i - b)}), \quad \text{with} \quad y_i = \sum_{j=1}^{N} W_{ij} x_j + \gamma_i$$
$$\tau_w \, \dot{W}_{ij} = D_{ij} x_i x_j (x_i - \theta_i) - \varepsilon W_{ij} \quad \text{with} \quad W_{ij}(0) = d_{ij} D_{ij}$$
$$\tau_\theta \, \dot{\theta}_i = x_i^2 - \theta_i.$$

θ_i : modify the threshold above (below) which the synapse potentiates (depresses).

$\alpha_i, \beta_i, \gamma_i$: distinguish the dynamical behavior of each node i.

$D = (D_{ij})_{i,j=1}^{N}$: structural backbone, $D_{ij} > 0$ if the presynaptic neuron j excites the postsynaptic neuron i, $D_{ij} < 0$ if the presynaptic neuron j inhibits the postsynaptic neuron i, and $D_{ij} = 0$ if no edge exist between neurons i and j.