NetSci 2024, Québec

Revisiting the Kuramoto model on graph:

integrals of motion, motifs, and symmetries

Vincent Thibeault, Benjamin Claveau, Antoine Allard, Patrick Desrosiers

20 juin 2024

Département de physique, de génie physique, et d'optique
Université Laval, Québec, Canada

Dynamica

CIMMUL
Centre Interdisciplinaire en Modélisation Mathématique de l'Université Laval

Université Laval

Kuramoto model on graph and mathematical contributions

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, \ldots, N\}$$

$\theta_j(t)$: j-th phase at time t

ω_j : j-th natural frequency

W : **real weight matrix**

σ : coupling constant

N : finite number of oscillators

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, ..., N\}$$

$\theta_j(t)$: j-th phase at time t

ω_j : j-th natural frequency

W : **real weight matrix**

σ : coupling constant

N : finite number of oscillators

1. We obtain the **necessary&sufficient** conditions on W for **partial integrability**.

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, \ldots, N\}$$

$\theta_j(t)$: j-th phase at time t

ω_j : j-th natural frequency

W : **real weight matrix**

σ : coupling constant

N : finite number of oscillators

1. We obtain the **necessary&sufficient** conditions on W for **partial integrability**.
2. We generalize **Watanabe-Strogatz reduction** to more heterogeneous graphs.

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, ..., N\}$$

$\theta_j(t)$: j-th phase at time t

ω_j : j-th natural frequency

W : **real weight matrix**

σ : coupling constant

N : finite number of oscillators

1. We obtain the **necessary&sufficient** conditions on W for **partial integrability**.
2. We generalize **Watanabe-Strogatz reduction** to more heterogeneous graphs.
3. We demystify the reducibility with **Lie symmetries** and **Koopman theory**.

With identical frequencies and a complete graph ($W_{jk} = 1$ for all j, k),

$$\dot{\theta}_j = \omega + \sigma \sum_{k=1}^{N} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, ..., N\}$$

With identical frequencies and a complete graph ($W_{jk} = 1$ for all j, k),

$$\dot{\theta}_j = \omega + \sigma \sum_{k=1}^{N} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, ..., N\}$$

- $N - 3$ integrals of motion (functions of $\theta_1, ..., \theta_N$ with null time derivative) ;

Watanabe, Strogatz, 1994 : Identical oscillators...non-trivial, but partially integrable !

With identical frequencies and a complete graph ($W_{jk} = 1$ for all j, k),

$$\dot{\theta}_j = \omega + \sigma \sum_{k=1}^{N} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, \ldots, N\}$$

- $N - 3$ integrals of motion (functions of $\theta_1, \ldots, \theta_N$ with null time derivative) ;
- *Exactly* reducible to 3 differential equations using

$$\tan((\theta_j(t) - \Theta(t))/2) = \sqrt{\frac{1 + \gamma(t)}{1 - \gamma(t)}}\, \tan((\psi_j - \Psi(t))/2) \qquad \text{(WS transformation)}$$

1. **Change of coordinates** : $z_j(t) = e^{i\theta_j(t)}$

1. **Change of coordinates** : $z_j(t) = e^{i\theta_j(t)}$

2. Kuramoto model as a system of **Riccati equations** :

$$\dot{z}_j(t) = i(f(t)z_j(t)^2 + \omega z_j(t) + \bar{f}(t)) \quad \text{with} \quad f(t) = \frac{i\sigma}{2}\sum_{k=1}^{N} \bar{z}_k(t)\,, \qquad \forall j$$

1. **Change of coordinates** : $z_j(t) = e^{i\theta_j(t)}$

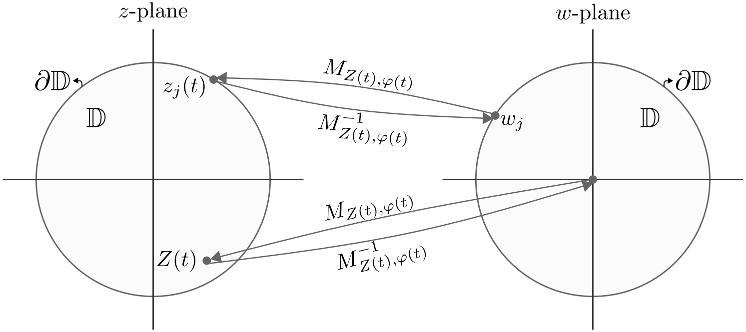
2. Kuramoto model as a system of **Riccati equations** :

$$\dot{z}_j(t) = i(f(t)z_j(t)^2 + \omega z_j(t) + \bar{f}(t)) \quad \text{with} \quad f(t) = \frac{i\sigma}{2}\sum_{k=1}^{N}\bar{z}_k(t)\,, \qquad \forall j$$

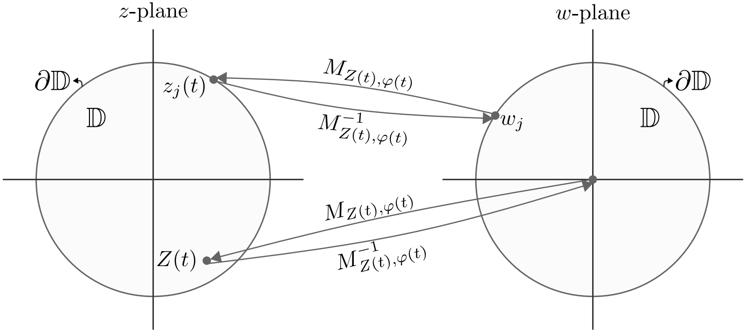
3. Solutions are **Möbius transformations** :

$$z_j(t) = \frac{y_j + F(t)}{y_j + G(t)}\,H(t)\,, \qquad \forall j\,.$$

Not any Mobius transformation : **automorphisms of the disk** (equivalent to WS)



Not any Mobius transformation : **automorphisms of the disk** (equivalent to WS)



The integrals of motions are **cross-ratios** (to remember !) :

$$c_{abcd}(z) = (z_a, z_b\,;\, z_c, z_d) = \frac{(z_c - z_a)(z_d - z_b)}{(z_c - z_b)(z_d - z_a)}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t} c_{abcd}(z(t)) = 0\,.$$

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,,\;\; j \in \{1, ..., N\}$$

Does the Watanabe-Strogatz approach fails when the graph is not complete?

1994

Physica D 74 (1994) 197–253

Constants of motion for superconducting Josephson arrays

Shinya Watanabe, Steven H. Strogatz
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Received 10 December 1993; revised 14 February 1994; accepted 14 February 1994

Consider N identical phase oscillators

$$\dot{\theta}_j = f + g\cos\theta_j + h\sin\theta_j, \qquad j = 1, \cdots, N,$$

The key restriction is that f, g, h must not depend on the subscript j.

2009

CHAOS **19**, 043104 (2009)

Identical phase oscillators with global sinusoidal coupling evolve by Möbius group action

Seth A. Marvel,[1,a)] Renato E. Mirollo,[2] and Steven H. Strogatz[1]
[1]*Center for Applied Mathematics, Cornell University, Ithaca, New York 14853, USA*
[2]*Department of Mathematics, Boston College, Chestnut Hill, Massachusetts 02167, USA*

Josephson junction arrays. The key is that f and g must be the same for all oscillators, and thus do *not* depend on the index j. We call such systems *sinusoidally coupled* because

2019

Systems of matrix Riccati equations, linear fractional transformations, partial integrability and synchronization

Cite as: J. Math. Phys. **60**, 072701 (2019); https://doi.org/10.1063/1.5085248
Submitted: 11 December 2018 . Accepted: 22 June 2019 . Published Online: 19 July 2019

M. A. Lohe

the network topology must be complete, i.e., we allow only global connectivity.

2020

REVIEW **Open Access**

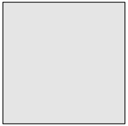
Understanding the dynamics of biological and neural oscillator networks through exact mean-field reductions: a review

Christian Bick[1,2,3,4,5*], Marc Goodfellow[2,3,6,7], Carlo R. Laing[8] and Erik A. Martens[9,10,11*]

The Ott–Antonsen reduction for the mean-field limit allows for the oscillators to be nonidentical. By contrast, the Watanabe–Strogatz reduction of finite networks requires oscillators to be identical. Neither of these approaches applies to finite networks of non-identical oscillators, and understanding such networks remains a challenge. Direct nu-

Short summary of existing results

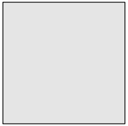
Watanabe, Strogatz, 1994

$W =$ 

integrals of motion : $N-3$

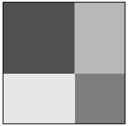
Short summary of existing results

Watanabe, Strogatz, 1994

$W =$ 

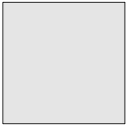
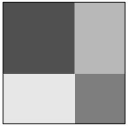
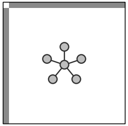
integrals of motion : $N-3$

Pikovsky, Rosenblum, 2008

$W =$ 

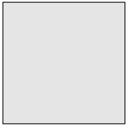
$N-3m$

Short summary of existing results

	Watanabe, Strogatz, 1994	Pikovsky, Rosenblum, 2008	Vlasov, Zou, Pereira, 2015
	$W =$ 	$W =$ 	$W =$ 
# integrals of motion:	$N-3$	$N-3m$	$(N-1)-3$

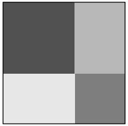
Short summary of existing results

Watanabe, Strogatz, 1994

$W =$ 

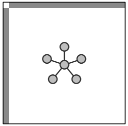
integrals of motion: $N-3$

Pikovsky, Rosenblum, 2008

$W =$ 

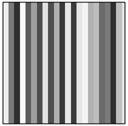
$N-3m$

Vlasov, Zou, Pereira, 2015

$W =$ 

$(N-1)-3$

Lohe, 2017

$W =$ 

$N-3$

What are the precise conditions on W so that a cross-ratio c_{abcd} is conserved?

Theorem

Let the Kuramoto model on a general graph with real weight matrix W be

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad\qquad j \in \{1, \dots, N\}\,.$$

A cross-ratio c_{abcd} is an integral of motion ***if and only if*** *vertices a,b,c,d have the same*

Theorem

Let the Kuramoto model on a general graph with real weight matrix W be

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, \dots, N\}\,.$$

A cross-ratio c_{abcd} is an integral of motion ***if and only if*** *vertices a,b,c,d have the same*

1. ***natural frequencies****, i.e.,* $\omega_a = \omega_b = \omega_c = \omega_d$;

Theorem

Let the Kuramoto model on a general graph with real weight matrix W be

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, ..., N\}\,.$$

*A cross-ratio c_{abcd} is an integral of motion **if and only if** vertices a,b,c,d have the same*

1. ***natural frequencies**, i.e., $\omega_a = \omega_b = \omega_c = \omega_d$;*
2. ***incoming edges** from vertices other than $\{a, b, c, d\}$, i.e.,*

$$W_{ak} = W_{bk} = W_{ck} = W_{dk}, \qquad \forall k \in \{1, ..., N\} \setminus \{a, b, c, d\};$$

Theorem

Let the Kuramoto model on a general graph with real weight matrix W be

$$\dot{\theta}_j = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j)\,, \qquad j \in \{1, \dots, N\}\,.$$

A cross-ratio c_{abcd} is an integral of motion ***if and only if*** *vertices a,b,c,d have the same*

1. ***natural frequencies****, i.e., $\omega_a = \omega_b = \omega_c = \omega_d$;*
2. ***incoming edges*** *from vertices other than $\{a, b, c, d\}$, i.e.,*

$$W_{ak} = W_{bk} = W_{ck} = W_{dk}, \qquad \forall k \in \{1, \dots, N\} \setminus \{a, b, c, d\};$$

3. ***outgoing edges*** *towards $\{a, b, c, d\}$, i.e.,*

$$W_{ba} = W_{ca} = W_{da}\,, \qquad W_{ac} = W_{bc} = W_{dc}\,,$$
$$W_{ab} = W_{cb} = W_{db}\,, \qquad W_{ad} = W_{bd} = W_{cd}\,.$$

Condition 3

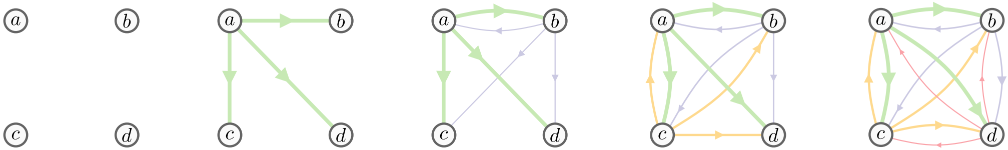
$$W_{ba} = W_{ca} = W_{da}\,, \qquad W_{ac} = W_{bc} = W_{dc}\,,$$
$$W_{ab} = W_{cb} = W_{db}\,, \qquad W_{ad} = W_{bd} = W_{cd}\,,$$

establishes all **motifs** of 4 vertices related to **one** integral of motion c_{abcd}.

Condition 3

$$W_{ba} = W_{ca} = W_{da}\,, \qquad W_{ac} = W_{bc} = W_{dc}\,,$$
$$W_{ab} = W_{cb} = W_{db}\,, \qquad W_{ad} = W_{bd} = W_{cd}\,,$$

establishes all **motifs** of 4 vertices related to **one** integral of motion c_{abcd}.



* Always under the assumption that $\omega_a = \omega_b = \omega_c = \omega_d$ (condition 1) is satisfied.

Condition 2

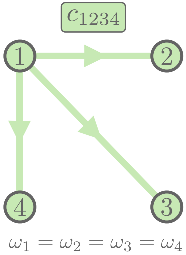
$$W_{ak} = W_{bk} = W_{ck} = W_{dk}, \qquad \forall k \in \{1, ..., N\} \setminus \{a, b, c, d\};$$

constrains **how to connect the motifs** to form a larger network.

Condition 2

$$W_{ak} = W_{bk} = W_{ck} = W_{dk}, \qquad \forall k \in \{1, ..., N\} \setminus \{a, b, c, d\};$$

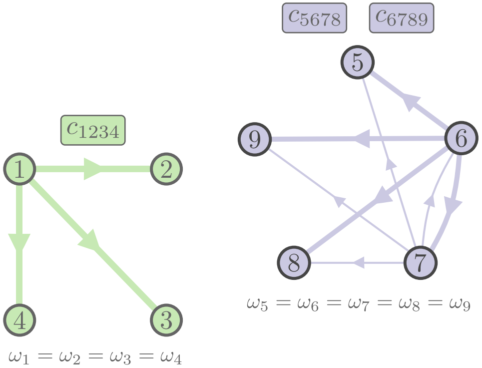
constrains **how to connect the motifs** to form a larger network.



Condition 2

$$W_{ak} = W_{bk} = W_{ck} = W_{dk}, \qquad \forall k \in \{1, ..., N\} \setminus \{a, b, c, d\};$$

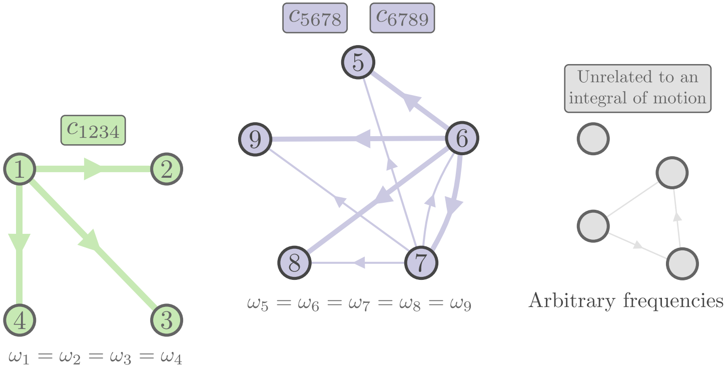
constrains **how to connect the motifs** to form a larger network.



Condition 2

$$W_{ak} = W_{bk} = W_{ck} = W_{dk}, \qquad \forall k \in \{1, ..., N\} \setminus \{a, b, c, d\};$$

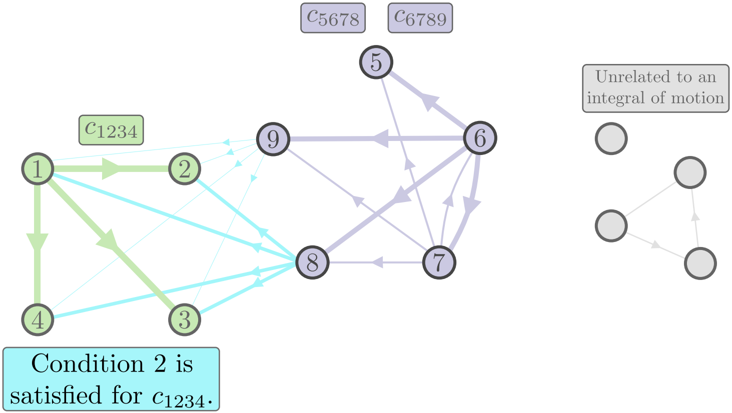
constrains **how to connect the motifs** to form a larger network.



Condition 2 becomes

$$W_{1k} = W_{2k} = W_{3k} = W_{4k}, \qquad \text{for } k = 8, 9 \notin \{1, 2, 3, 4\};$$

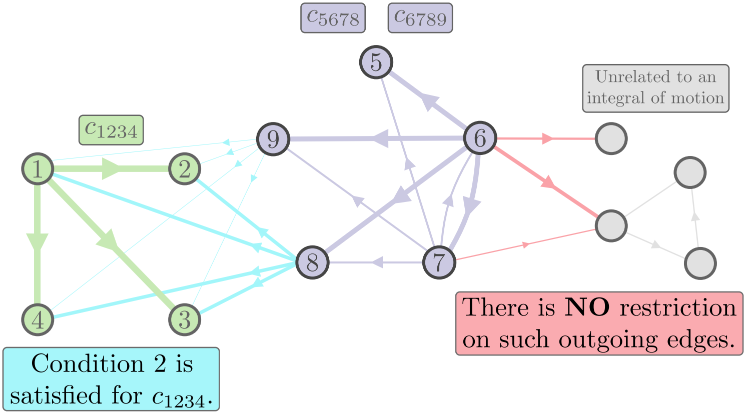
and constrains the connections from vertices $\{8, 9\}$ to vertices $\{1, 2, 3, 4\}$.



Condition 2 becomes

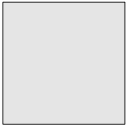
$$W_{1k} = W_{2k} = W_{3k} = W_{4k}, \qquad \text{for } k = 8, 9 \notin \{1,2,3,4\};$$

and constrains the connections from vertices $\{8, 9\}$ to vertices $\{1, 2, 3, 4\}$.



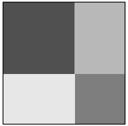
Generally, the oscillators have very heterogeneous **out-degrees**.

Watanabe, Strogatz, 1994

$W =$ 

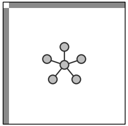
\# integrals of motion : $N - 3$

Pikovsky, Rosenblum, 2008

$W =$ 

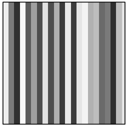
$N - 3m$

Vlasov, Zou, Pereira, 2015

$W =$ 

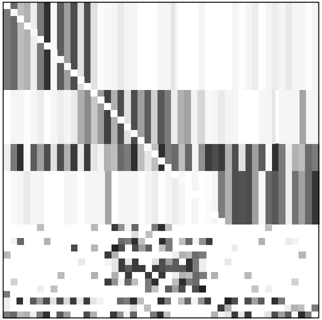
$(N - 1) - 3$

Lohe, 2017

$W =$ 

$N - 3$

Most general case from the theorem

$W =$ 

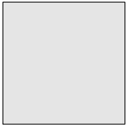
$N - n$ integrals of motion

$n \in \{3, ..., N\}$

Special class of networks :

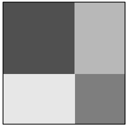
- Heterogeneous blocks ;

Watanabe, Strogatz, 1994

$W =$ 

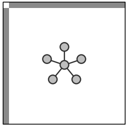
\# integrals of motion : $N - 3$

Pikovsky, Rosenblum, 2008

$W =$ 

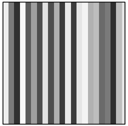
$N - 3m$

Vlasov, Zou, Pereira, 2015

$W =$ 

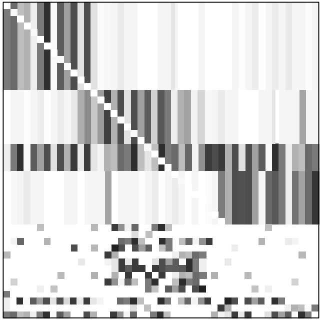
$(N-1) - 3$

Lohe, 2017

$W =$ 

$N - 3$

Most general case from the theorem

$W =$ 

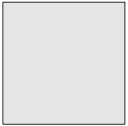
$N - n$ integrals of motion

$n \in \{3, ..., N\}$

Special class of networks :

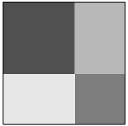
- Heterogeneous blocks ;
- Various motifs (not only stars) ;

Watanabe, Strogatz, 1994

$W =$ 

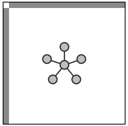
integrals of motion : $N - 3$

Pikovsky, Rosenblum, 2008

$W =$ 

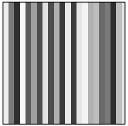
$N - 3m$

Vlasov, Zou, Pereira, 2015

$W =$ 

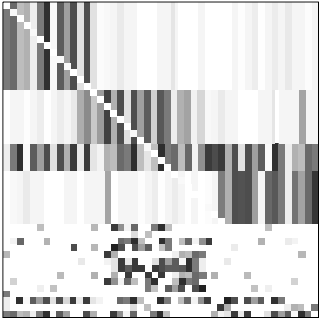
$(N - 1) - 3$

Lohe, 2017

$W =$ 

$N - 3$

Most general case from the theorem

$W =$ 

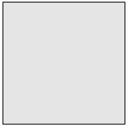
$N - n$ integrals of motion

$n \in \{3, ..., N\}$

Special class of networks :

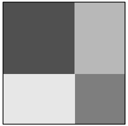
- Heterogeneous blocks ;
- Various motifs (not only stars) ;
- Unrestricted block ;

Watanabe, Strogatz, 1994

$W =$ 

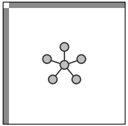
\# integrals of motion : $N - 3$

Pikovsky, Rosenblum, 2008

$W =$ 

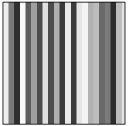
$N - 3m$

Vlasov, Zou, Pereira, 2015

$W =$ 

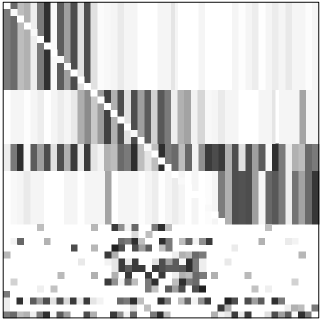
$(N - 1) - 3$

Lohe, 2017

$W =$ 

$N - 3$

Most general case from the theorem

$W =$ 

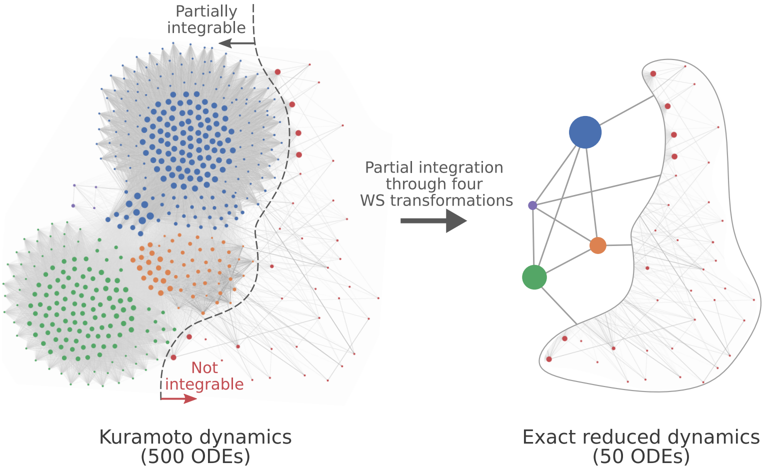
$N - n$ integrals of motion

$n \in \{3, ..., N\}$

Special class of networks :

- Heterogeneous blocks ;
- Various motifs (not only stars) ;
- Unrestricted block ;
- Weighted, directed, signed.

Generalization of the Watanabe-Strogatz reduction in one picture



Reduced system of $n = \#P_0 + 3m$ real equations.

If W contains m partially integrable parts, then in general [1] the number of reduced equation is thus

$$n = \mathrm{rank}(W) + 2m\,.$$

[1] ...under very mild assumptions...

If W contains m partially integrable parts, then in general [1] the number of reduced equation is thus

$$n = \mathrm{rank}(W) + 2m \,.$$

$\therefore$ A **low-rank hypothesis** on W implies that the **dynamics live in a low-dimensional space**.

nature physics

Article https://doi.org/10.1038/s41567-023-02303-0

The low-rank hypothesis of complex systems

Received: 18 October 2022

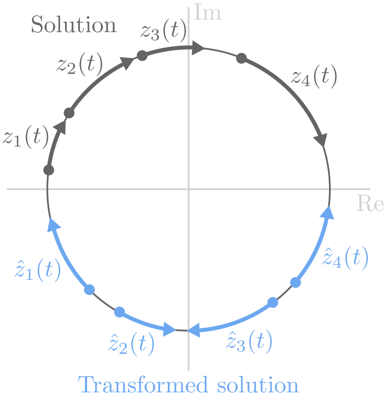
Accepted: 24 October 2023

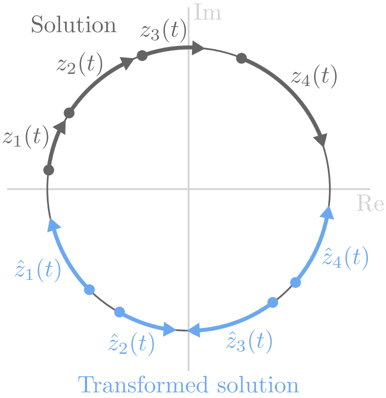
Published online: 10 January 2024

Vincent Thibeault [1,2], Antoine Allard [1,2] & Patrick Desrosiers [1,2,3]

Complex systems are high-dimensional nonlinear dynamical systems

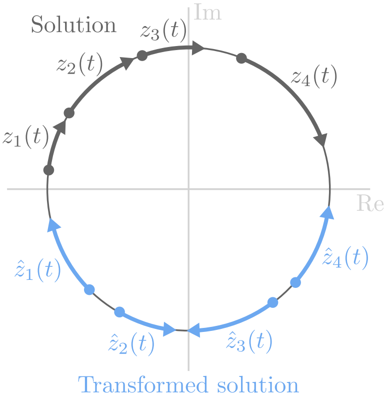
[1] ...under very mild assumptions...

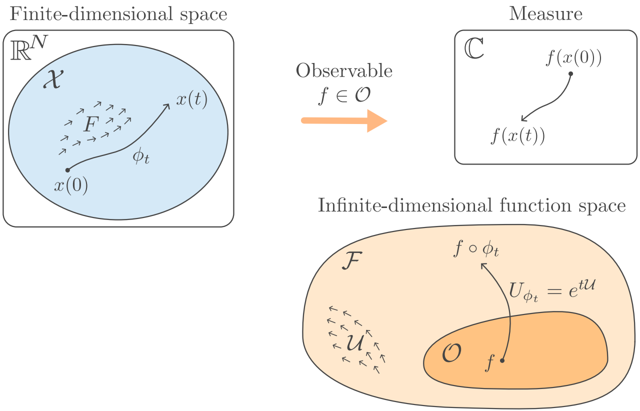

Solution
Im
Re
$z_1(t)$
$z_2(t)$
$z_3(t)$
$z_4(t)$
$\hat{z}_1(t)$
$\hat{z}_2(t)$
$\hat{z}_3(t)$
$\hat{z}_4(t)$
Transformed solution



We have one way to show that **there is a Lie symmetry group** acting behind the scenes.

Why it works? Lie symmetry group and Koopman theory (teaser)





We have one way to show that **there is a Lie symmetry group** acting behind the scenes.

Integrals of motions : $\mathcal{U}\, c_{abcd} = 0$

Symmetry condition : $[\mathcal{U}, v] = 0$

Take-home message and acknowledgments

- We proved the necessary and sufficient conditions for partial integrability in the Kuramoto model on graph ;

- We proved the necessary and sufficient conditions for partial integrability in the Kuramoto model on graph ;
- We generalized Watanabe-Strogatz reduction and showed how to design the graph to have an arbitrary number of integrals of motion from 0 to $N-3$.

Take-home message and acknowledgments

- We proved the necessary and sufficient conditions for partial integrability in the Kuramoto model on graph ;
- We generalized Watanabe-Strogatz reduction and showed how to design the graph to have an arbitrary number of integrals of motion from 0 to $N-3$.

Partial integrability is **not restricted** to identically connected Kuramoto oscillators !

- We proved the necessary and sufficient conditions for partial integrability in the Kuramoto model on graph ;
- We generalized Watanabe-Strogatz reduction and showed how to design the graph to have an arbitrary number of integrals of motion from 0 to $N-3$.

Partial integrability is **not restricted** to identically connected Kuramoto oscillators !

References : Pikovsky, Rosenblum, *Phys. Rev. Lett.*, 2008
Marvel, Mirollo, Strogatz, *Chaos*, 2009
Lohe, *J. Math. Phys.*, 2019
Thibeault, Allard, Desrosiers, *Nat. Phys.*, 2024

Contact : vincent.thibeault.1@ulaval.ca

Thanks to my collaborators and thank you for your attention ! Questions ?

Fonds de recherche Nature et technologies Québec

CRSNG NSERC

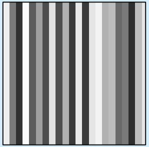
Sentinelle Nord

Conditions such that there is a **maximal number** $N-3$ of integrals of motion?

COROLLARY (INFORMAL)

The maximal number of integrals of motion is attained if (1) the frequencies are identical and (2) the weight matrix has the form

Lohe, 2017

$W =$ 

Graph automorphism ?

Graph automorphisms require permuting the rows AND the columns of the weight matrix W and preserve W, which is not the case in general.

The **absence of restriction on self-loops** in the theorem is a consequence of the sinusoidal interaction :

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \omega_j + \sigma \sum_{k=1}^{N} W_{jk} \sin(\theta_k - \theta_j),$$

implying that the self-loops have **no impact** on the dynamics.

Generalization of Watanabe-Strogatz reduction

1. Make the change of variable $z_j = e^{i\theta_j}$:

$$\dot{z}_j = i\omega_j z_j + p_{j,1} - p_{j,-1} z_j^2 \,, \qquad p_{j,n} = \frac{\sigma}{2} \sum_{k=1}^{N} W_{jk} z_k^n \,.$$

2. Partition the graph as $\mathcal{P} = \{P_0, P_1, ..., P_m\}$ where
   - P_0 : non-partially integrable part ;
   - $P_1, ..., P_m$: partially integrable parts .

3. Apply the conditions of the theorem for $P_1, ..., P_m$;

4. Apply different Möbius transformations for each $P_1, ..., P_m$;

5. Obtain the reduced system ;

$$\dot{Z}_\mu = i\Omega_\mu Z_\mu + p_{1,\mu} - p_{-1,\mu} Z_\mu^2\,, \qquad \mu \in \{1, ..., m\}$$

$$\dot{\varphi}_\mu = \Omega_\mu - 2\,\mathrm{Im}\,[p_{-1,\mu} Z_\mu]$$

$$\dot{z}_j = i\omega_j z_j + \frac{\sigma}{2}\sum_{k\in P_0} W_{jk}(z_k - z_j^2\bar{z}_k) + \frac{\sigma}{2}\sum_{k\in P_{\mathrm{I}}} W_{jk}[\zeta_{s(k)k} - z_j^2\,\overline{\zeta}_{s(k)k}]\,, \qquad j \in P_0$$

where

$$p_{n,\mu} = \frac{\sigma}{2}\sum_{k\in P_0} \mathcal{W}_{\mu k} z_k^n + \frac{\sigma}{2}\sum_{k\in P_{\mathrm{I}}} \mathcal{W}_{\mu k}\zeta_{\mu k}^n \quad \text{and} \quad \zeta_{\mu\ell} = M_{Z_\mu,\varphi_\mu}(b_\ell)\,.$$

There are

$$n = \#P_0 + 3m$$

real equations in the reduced system.