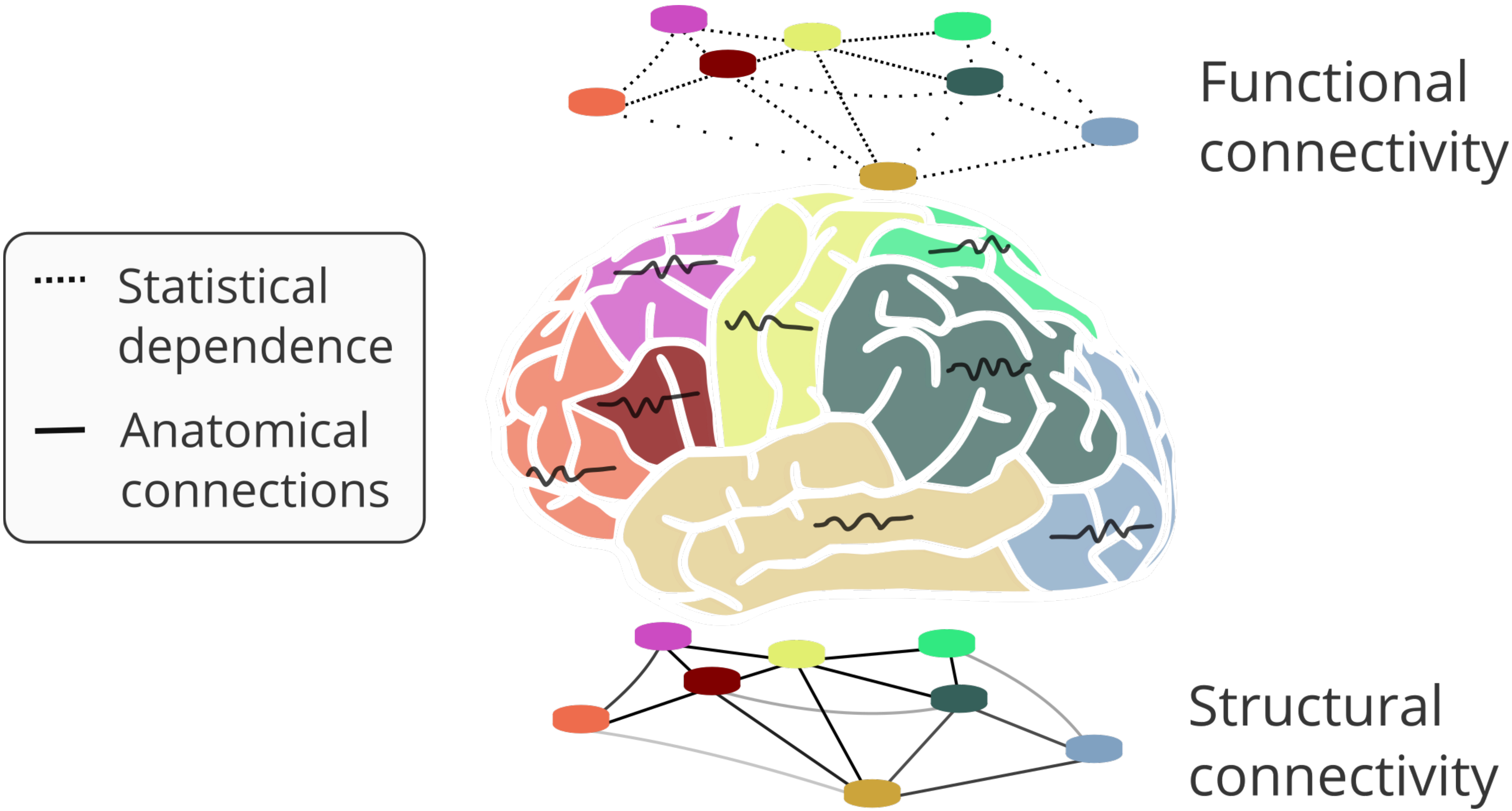
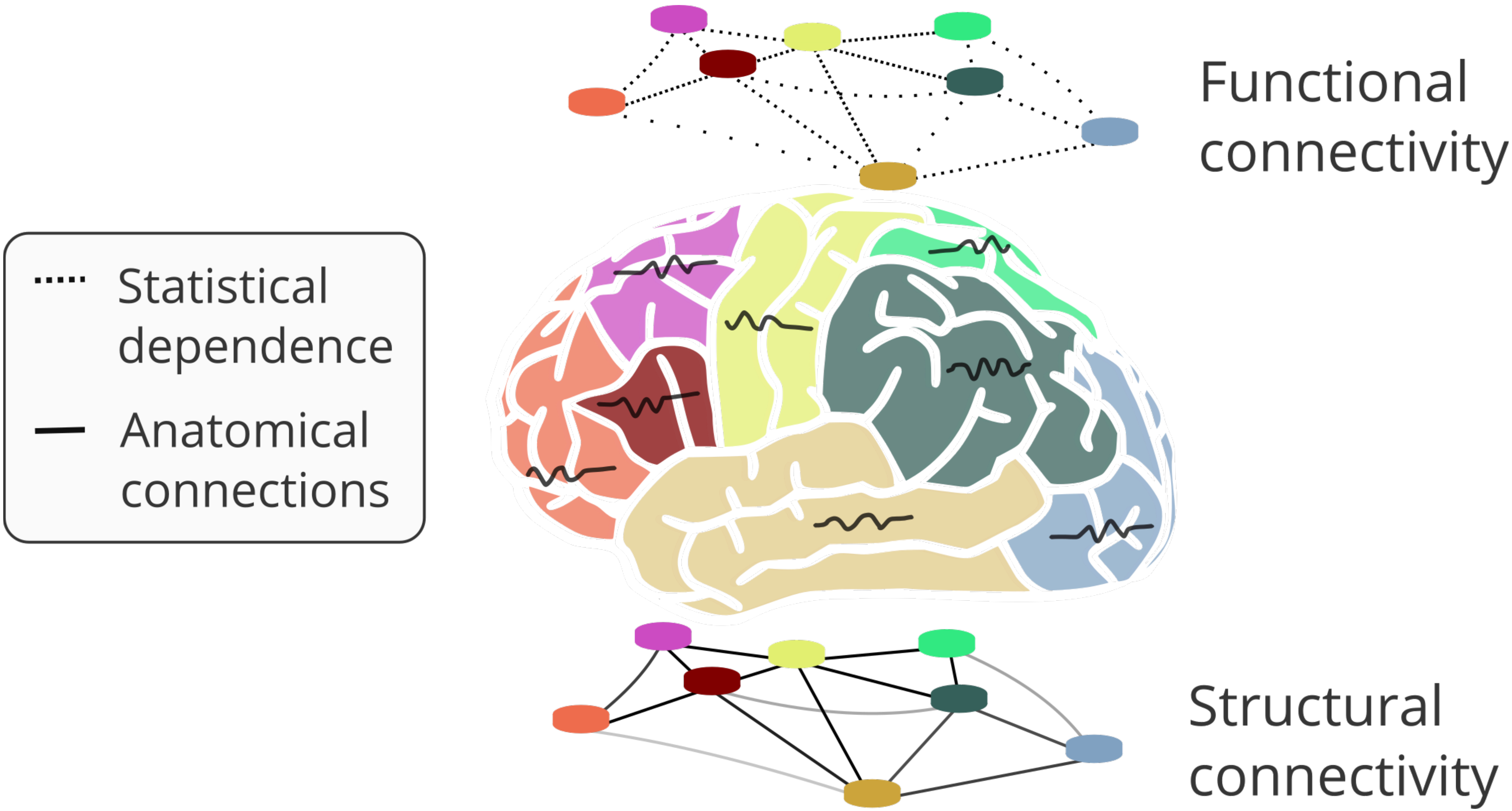
Analytical prediction of structure-function relationships in networks of Kuramoto oscillators

Arthur Légaré, Nicolas Doyon & Patrick Desrosiers

June 4, 2026


Statistical dependence
Anatomical connections
Functional connectivity
Structural connectivity



Can we predict functional from structural connectivity?

Method: coupled phase oscillators

We utilize the networked **Kuramoto model** to describe the state $\boldsymbol{\theta} \in \mathbb{R}^N$, governed by

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \omega_j + \frac{\lambda}{N} \sum_{l=1}^{N} K_{jl} \sin(\theta_l - \theta_j) \quad j \in \{1, \ldots, N\}$$

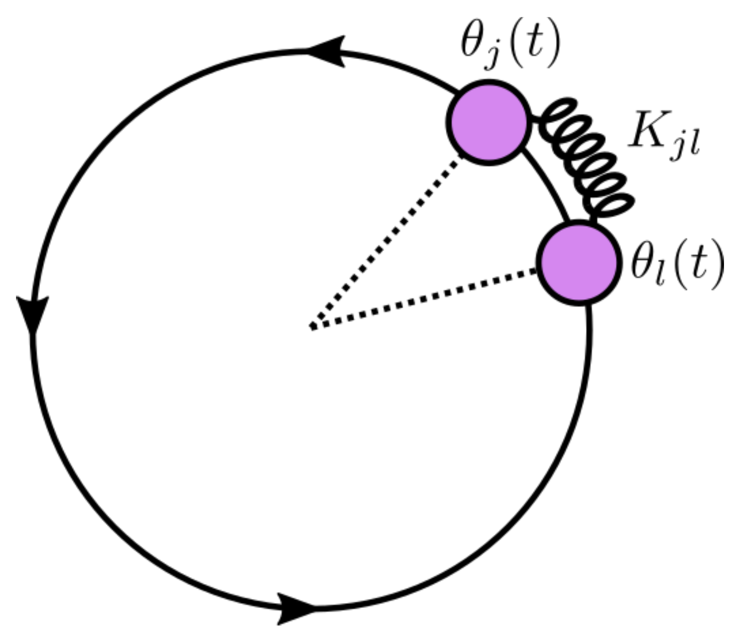
Method: coupled phase oscillators

We utilize the networked **Kuramoto model** to describe the state $\boldsymbol{\theta} \in \mathbb{R}^N$, governed by

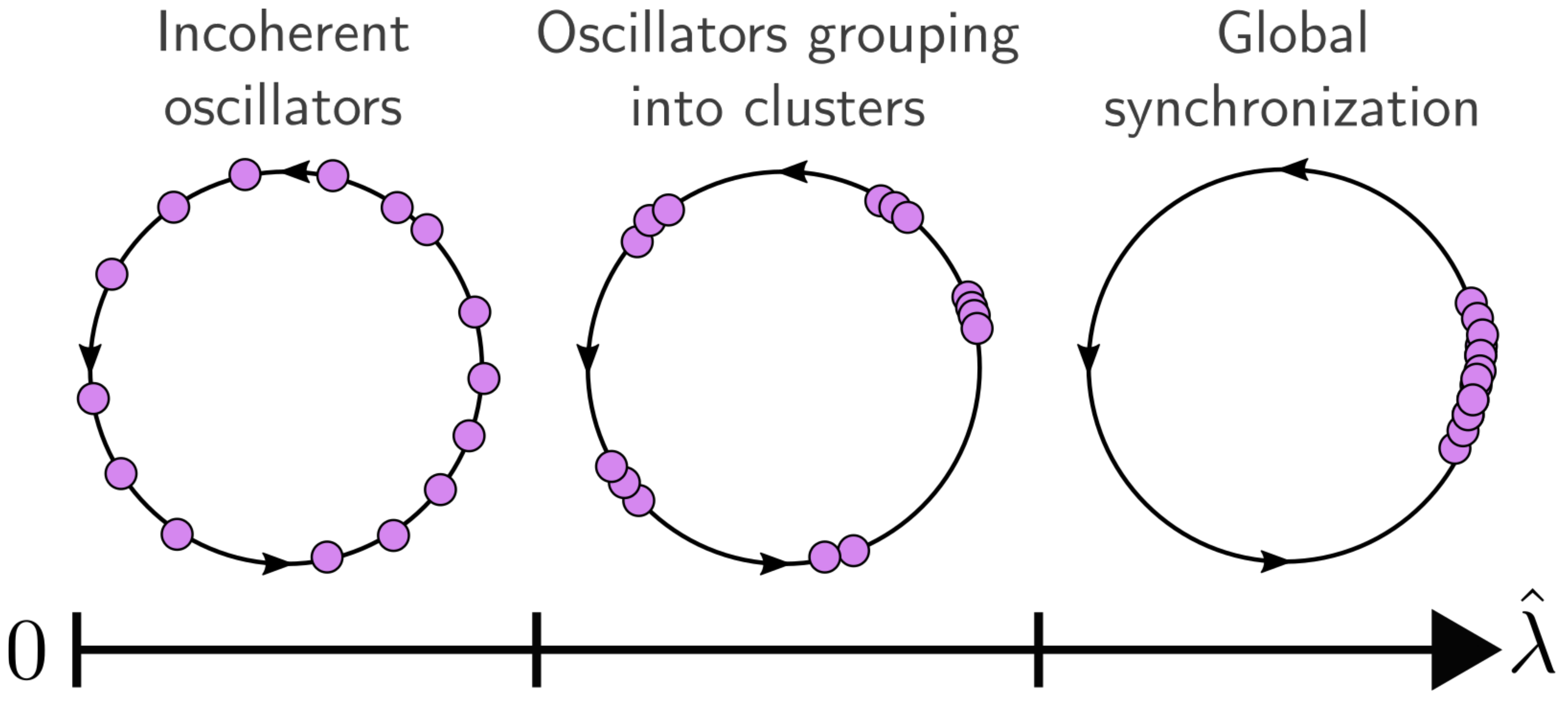
$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \omega_j + \frac{\lambda}{N} \sum_{l=1}^{N} K_{jl} \sin(\theta_l - \theta_j) \quad j \in \{1, \ldots, N\}$$

where:

- $\theta_j(t) \in \mathbb{R}$: phase of oscillator j at time t;
- $\omega_j \in \mathbb{R}$: natural frequency of oscillator j;
- $\frac{\lambda}{N} =: \hat{\lambda} \in \mathbb{R}_{\geq 0}$: global coupling strength;
- $K_{jl} \in \mathbb{R}$: structural connectivity $j \leftarrow l$.

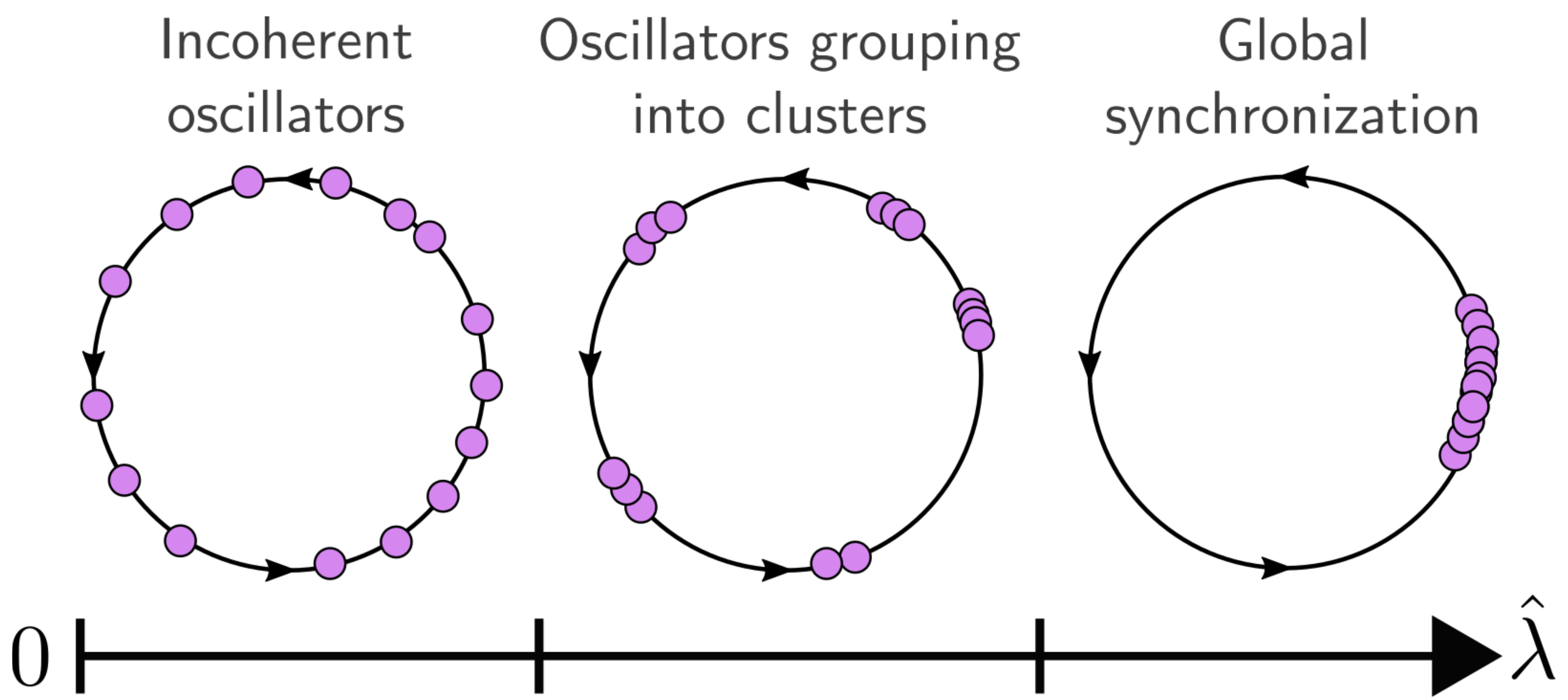


Dynamical regime of interest – Iso-Phase Ansatz (IPA)



[1] S. Strogatz (2000), *Physica D.*
[2] A. Arenas et al. (2006), *Phys. Rev. Lett.*
[3] M. Brede (2008), *Eur. Phys. J.*

Dynamical regime of interest – Iso-Phase Ansatz (IPA)



Frequency dispersion dominates.

Heterogenous K drives local phase-locking.

Oscillators all collapse into phase-locking.

[1] S. Strogatz (2000), *Physica D.*
[2] A. Arenas et al. (2006), *Phys. Rev. Lett.*
[3] M. Brede (2008), *Eur. Phys. J.*

Analytical roadmap (overview)

0. Correlation as a measure of functional connectivity

$$C_{jk}(T) = \frac{2}{T}\int_0^T \sin\theta_j(t)\sin\theta_k(t)\mathrm{d}t$$

Analytical roadmap (overview)

0. Correlation as a measure of functional connectivity

$$C_{jk}(T) = \frac{2}{T}\int_0^T \sin\theta_j(t)\sin\theta_k(t)\mathrm{d}t$$

1. Perturbative approx. of phases

$$\theta_j(t) \approx \theta_j^{(0)}(t) + \hat{\lambda}\theta_j^{(1)}(t) + \hat{\lambda}^2\theta_j^{(2)}(t)$$

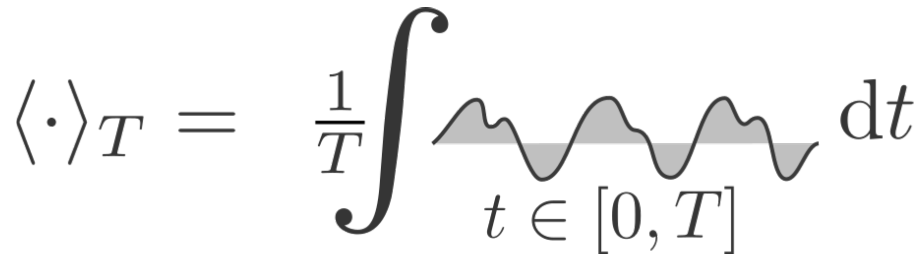
2. Perturbative approx. of coactivity

$$c_{jk}(t) \approx c_{jk}^{(0)}(t) + \hat{\lambda}c_{jk}^{(1)}(t) + \hat{\lambda}^2 c_{jk}^{(2)}$$

Analytical roadmap (overview)

3. Time-average of coactivity (correlation)

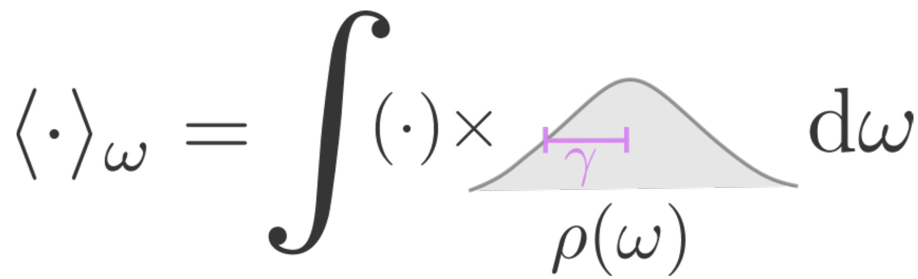
$$C_{jk}(T;\boldsymbol{\omega}) = \langle 2 \times c_{jk}(t;\boldsymbol{\omega}) \rangle_T$$



T : integration time

4. Ensemble-average over natural frequencies

$$\hat{C}_{jk}(T) = \langle C_{jk}(T;\boldsymbol{\omega}) \rangle_{\boldsymbol{\omega}}$$

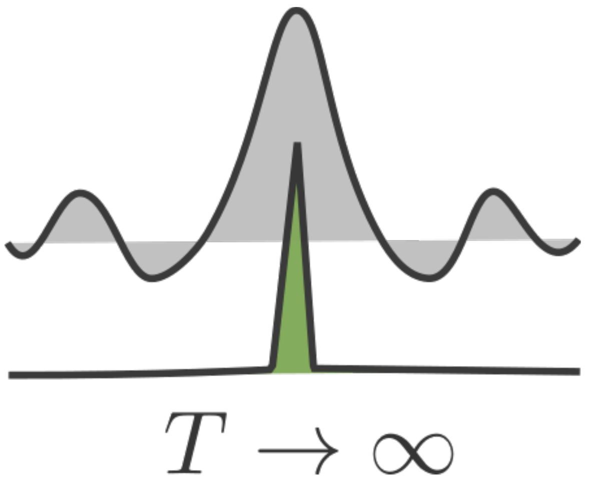


γ : dispersion of natural frequencies

Analytical roadmap (overview)

5. Stationary limit — identification of non-vanishing candidates

$$\hat{C}_{jk} = \lim_{T\to\infty} \hat{C}_{jk}(T)$$



Analytical roadmap

1. Perturbative expansion in powers of $\hat{\lambda} = \lambda / N$

Phase trajectory approximation

$$\theta_j(t) \approx \theta_j^{(0)}(t) + \hat{\lambda}\theta_j^{(1)}(t) + \hat{\lambda}^2\theta_j^{(2)}(t)$$

Analytical roadmap

1. Perturbative expansion in powers of $\hat{\lambda} = \lambda/N$

Phase trajectory approximation

$$\theta_j(t) \approx \theta_j^{(0)}(t) + \hat{\lambda}\theta_j^{(1)}(t) + \hat{\lambda}^2\theta_j^{(2)}(t)$$

initialized with $\theta_j^{(0)}(t) = \omega_j t$ is obtained from the Volterra integral form of Kuramoto

$$\theta_j(t) = \omega_j t + \hat{\lambda} \int_0^t \sum_{l=1}^{N} K_{jl} \sin(\theta_l(s) - \theta_j(s))\mathrm{d}s.$$

Analytical roadmap

2. Perturbative expansion of coactivity (integrand of corr. function)

Coactivity approximation

$$c_{jk}(t) := \sin\theta_j(t)\sin\theta_k(t) \approx c_{jk}^{(0)}(t) + \hat{\lambda} c_{jk}^{(1)}(t) + \hat{\lambda}^2 c_{jk}^{(2)}(t)$$

is obtained by inserting perturbative expressions for phase trajectories:

Analytical roadmap

2. Perturbative expansion of coactivity (integrand of corr. function)

Coactivity approximation

$$c_{jk}(t) := \sin\theta_j(t)\sin\theta_k(t) \approx c_{jk}^{(0)}(t) + \hat{\lambda} c_{jk}^{(1)}(t) + \hat{\lambda}^2 c_{jk}^{(2)}(t)$$

is obtained by inserting perturbative expressions for phase trajectories:

$$\begin{aligned} c_{jk}(t) \approx\ & \sin\theta_j^{(0)} \sin\theta_k^{(0)} \\ & +\hat{\lambda}\left[\theta_j^{(1)} \cos\theta_j^{(0)} \sin\theta_k^{(0)} + \theta_k^{(1)} \sin\theta_j^{(0)} \cos\theta_k^{(0)}\right] \\ & +\hat{\lambda}^2\left[\theta_j^{(1)}\theta_k^{(1)} \cos\theta_j^{(0)} \cos\theta_k^{(0)}\right. \\ & \quad -\tfrac{1}{2}\left((\theta_j^{(1)})^2 + (\theta_k^{(1)})^2\right) \sin\theta_j^{(0)} \sin\theta_k^{(0)} \\ & \quad \left.+\theta_j^{(2)} \cos\theta_j^{(0)} \sin\theta_k^{(0)} + \theta_k^{(2)} \sin\theta_j^{(0)} \cos\theta_k^{(0)}\right]. \end{aligned}$$

Analytical roadmap

3. Correlation by time-averaging coactivity

Up to $\mathcal{O}(\hat{\lambda}^2)$, many different matrix forms are obtained:

$$
\begin{aligned}
C(T;\boldsymbol{\omega}) = I &+ \hat{\lambda}\alpha^{(1)}(T;\boldsymbol{\omega})[JK + K^\intercal J] \circ \overline{J} \\
&+ \hat{\lambda}^2\alpha_a^{(2)}(T;\boldsymbol{\omega})[K^2J + J(K^2)^\intercal] \circ \overline{J} \\
&+ \hat{\lambda}^2\alpha_b^{(2)}(T;\boldsymbol{\omega})[(KJ)^{\circ 2} + (JK^\intercal)^{\circ 2}] \circ \overline{J} \\
&+ \hat{\lambda}^2\alpha_c^{(2)}(T;\boldsymbol{\omega})[K^{\circ 2}J + J(K^{\circ 2})^\intercal] \circ \overline{J} \\
&+ \hat{\lambda}^2\alpha_d^{(2)}(T;\boldsymbol{\omega})[K\overline{J}K^\intercal] \circ \overline{J} \\
&+ \hat{\lambda}^2\alpha_e^{(2)}(T;\boldsymbol{\omega})[KK^\intercal] \circ \overline{J},
\end{aligned}
$$

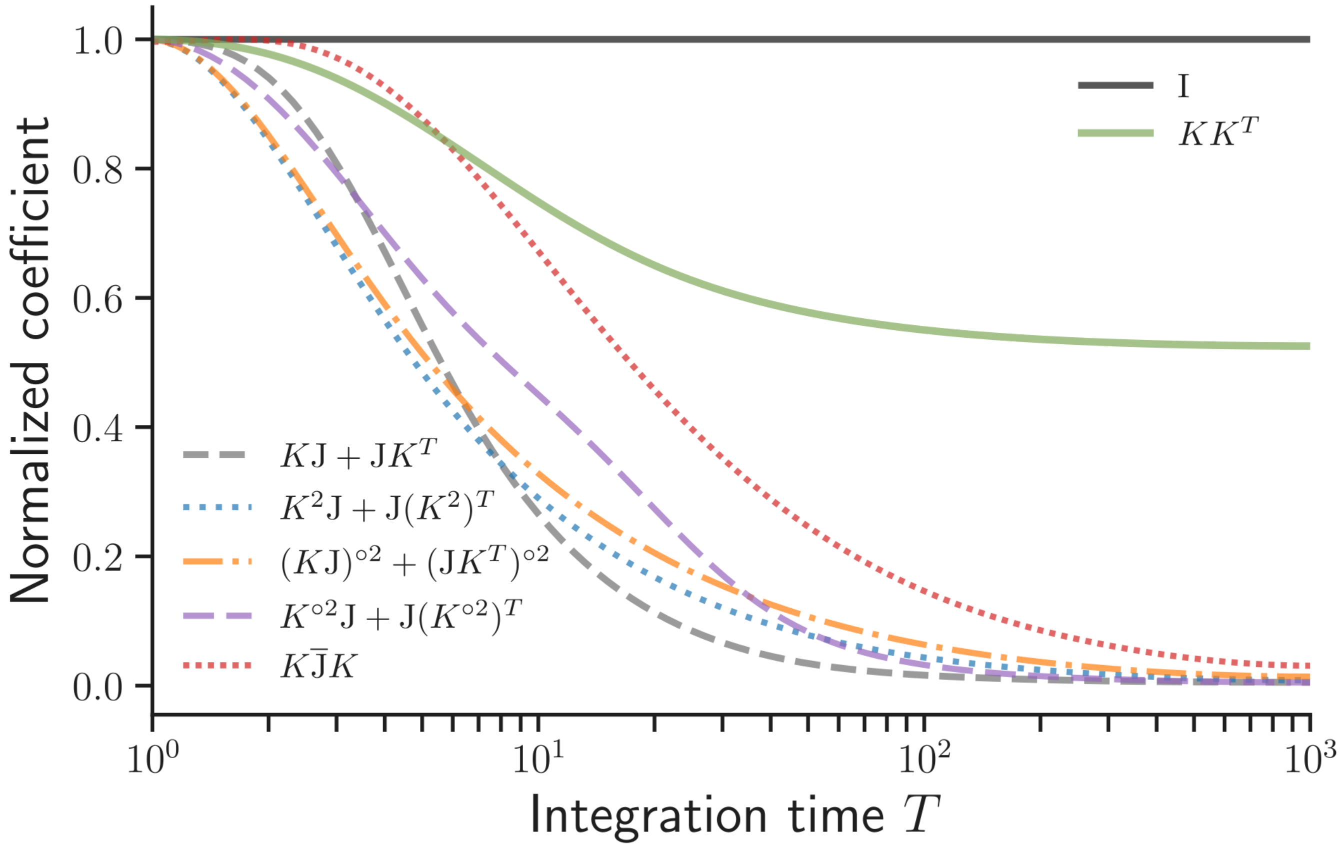
where $J = \mathbf{1}\mathbf{1}^\intercal$ is the matrix of ones, $\overline{J} = J - I$ its off-diagonal counterpart, and "$\circ$" denotes the element-wise operation.

Analytical roadmap

4. Ensemble average over natural frequencies

$$\langle \cdot \rangle_\omega = \int (\cdot) \times \rho(\omega)\, \mathrm{d}\omega$$

γ : dispersion of natural frequencies



5. Stationary limit

Dimensional analysis yields the power-counting criterion

$$\eta := s - m + 1 \begin{cases} = 0 \implies \text{stationary} \\ < 0 \implies \text{vanishing} \end{cases}$$

where, for a given α function,

- s : number of resonant divisors,
- m : dimension of space-frequency.

5. Stationary limit

Dimensional analysis yields the power-counting criterion

$$\eta := s - m + 1 \begin{cases} = 0 \implies \text{stationary} \\ < 0 \implies \text{vanishing} \end{cases}$$

where, for a given α function,

- s : number of resonant divisors,
- m : dimension of space-frequency.

From the > 100 integrals generated by the perturbative expansion, now < 10 need to be solved. This yields...

Analytical solution for functional connectivity

$$\boxed{\hat{C}(\gamma, \hat{\lambda}, K) = I + \frac{5\hat{\lambda}^2}{4\gamma^2}\Big(KK^\top - \mathrm{diag}(KK^\top)\Big)}$$

Analytical solution for functional connectivity

$$\hat{C}(\gamma, \hat{\lambda}, K) = I + \frac{5\hat{\lambda}^2}{4\gamma^2}\left(KK^\top - \mathrm{diag}(KK^\top)\right)$$

- Absence of direct structural connections (K) in functional connectivity ($\hat{C}$)

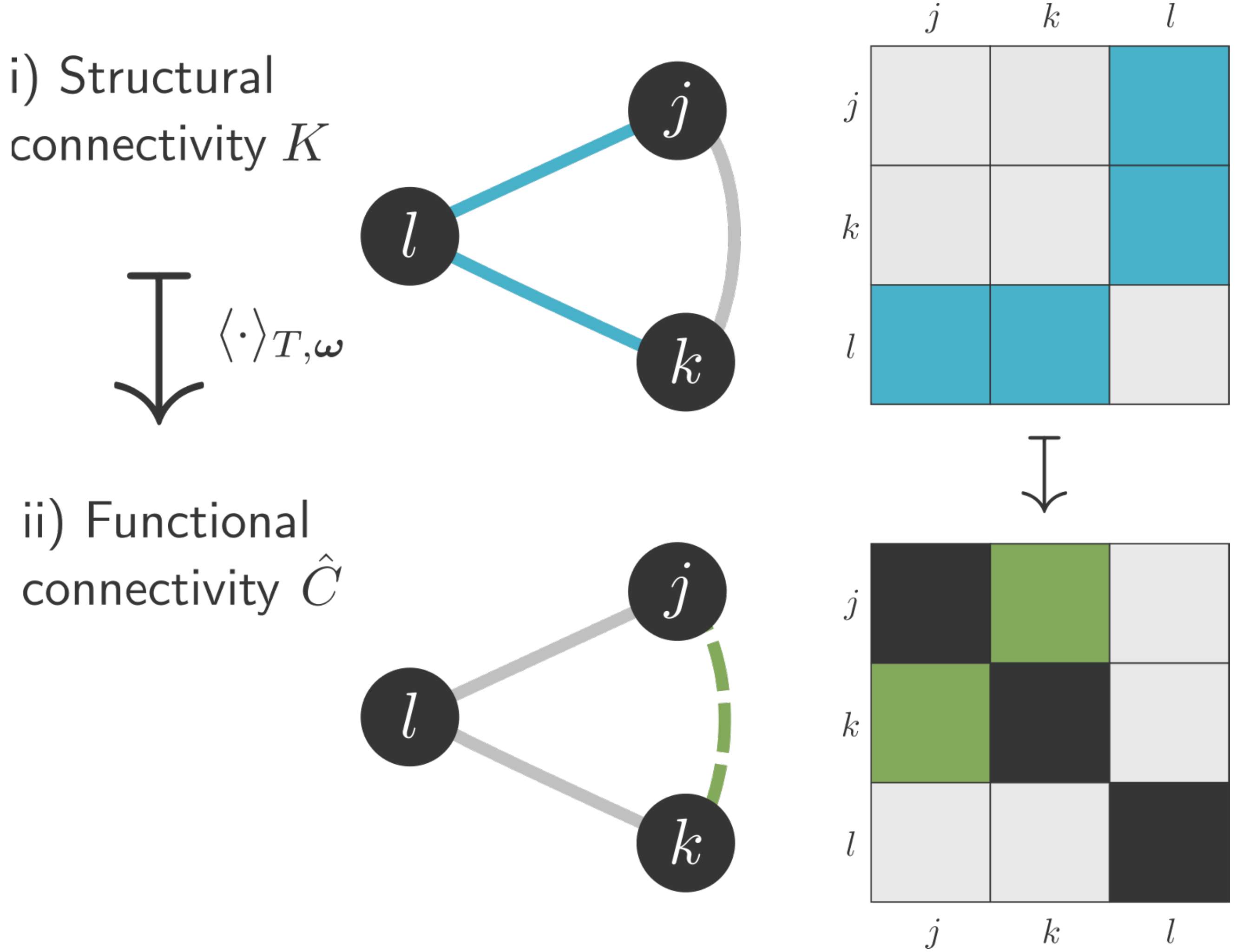
Analytical solution for functional connectivity

$$\hat{C}(\gamma, \hat{\lambda}, K) = I + \frac{5\hat{\lambda}^2}{4\gamma^2}\left(KK^\top - \mathrm{diag}(KK^\top)\right)$$

- Absence of direct structural connections (K) in functional connectivity ($\hat{C}$)
- **Dispersion**–**coupling** linear tradeoff

Analytical solution for functional connectivity

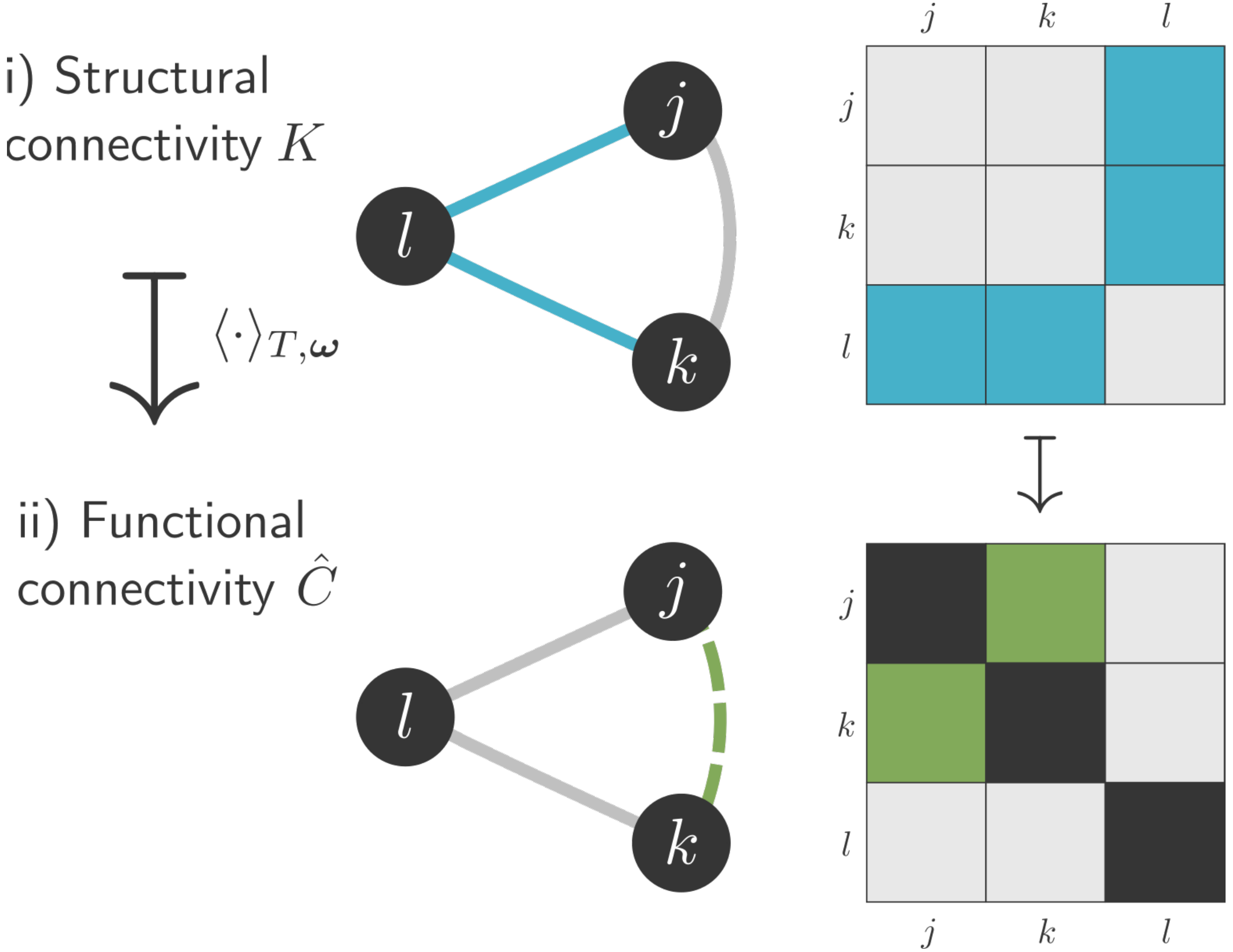
$$\hat{C}(\gamma, \hat{\lambda}, K) = I + \frac{5\hat{\lambda}^2}{4\gamma^2}\left(KK^\top - \mathrm{diag}(KK^\top)\right)$$



Analytical solution for functional connectivity

$$\hat{C}(\gamma, \hat{\lambda}, K) = I + \frac{5\hat{\lambda}^2}{4\gamma^2}\left(KK^\top - \mathrm{diag}(KK^\top)\right)$$

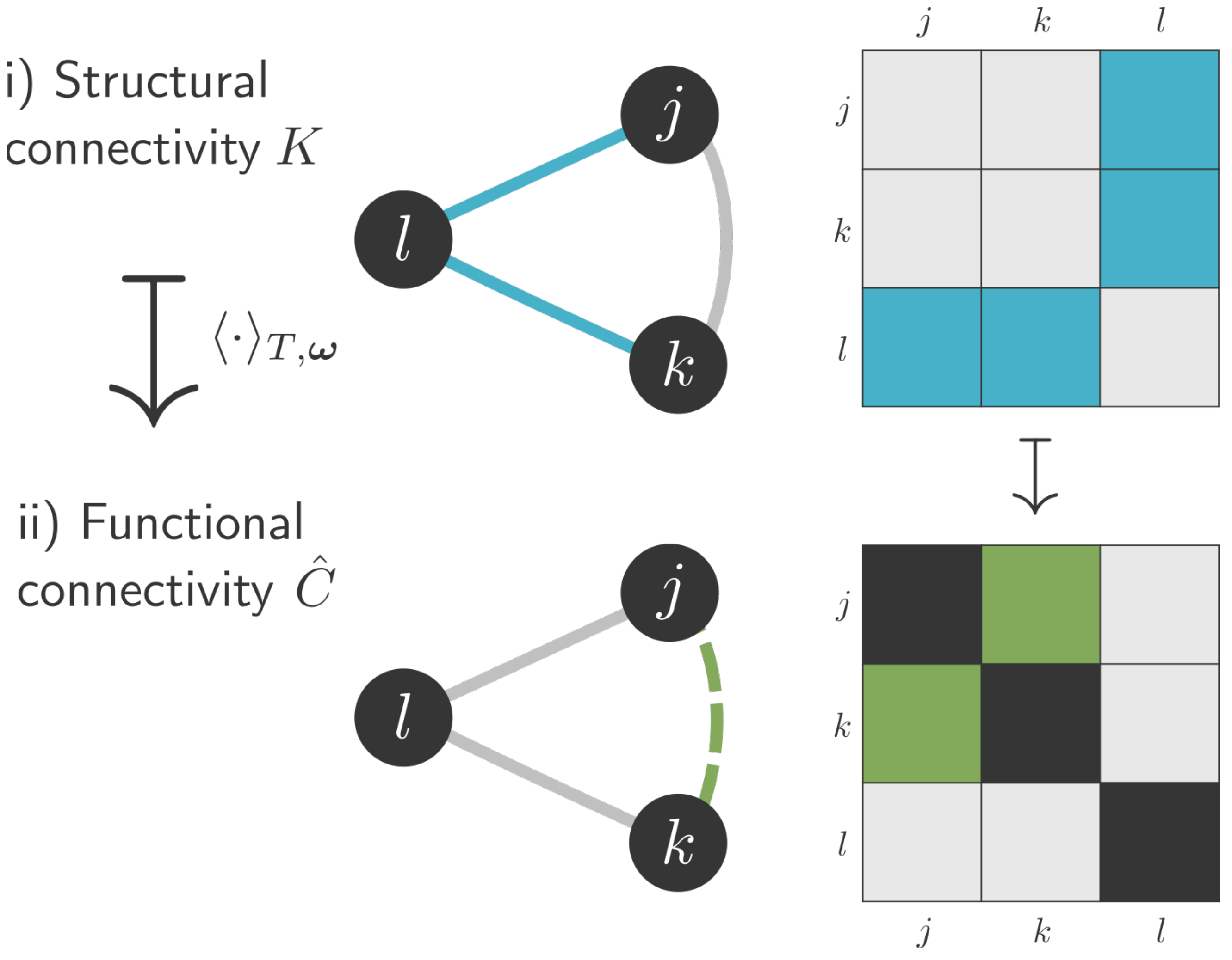
- Nodes j and k correlated through a common neighbor l



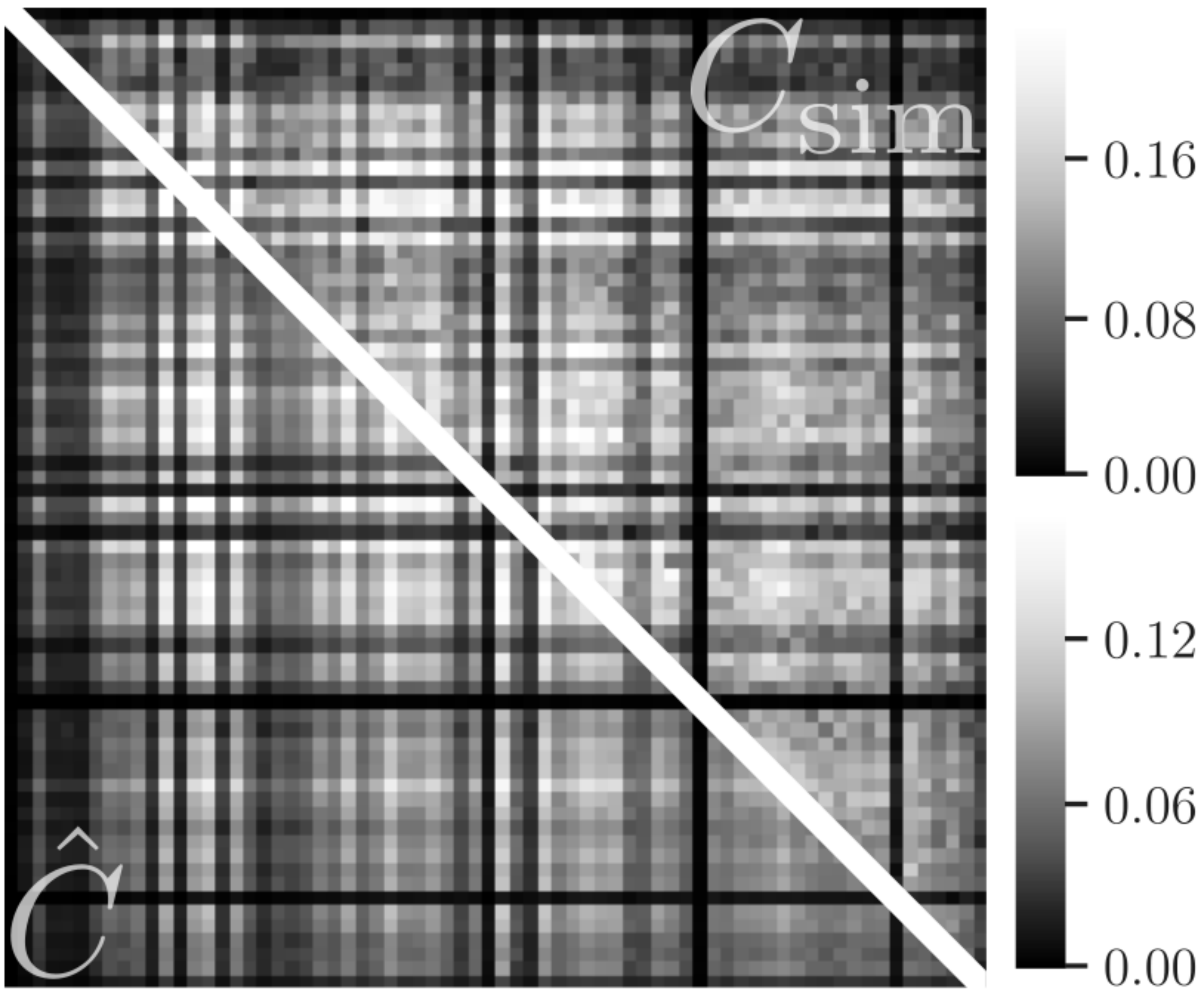
Analytical solution for functional connectivity

$$\hat{C}(\gamma, \hat{\lambda}, K) = I + \frac{5\hat{\lambda}^2}{4\gamma^2}\left(KK^\top - \mathrm{diag}(KK^\top)\right)$$

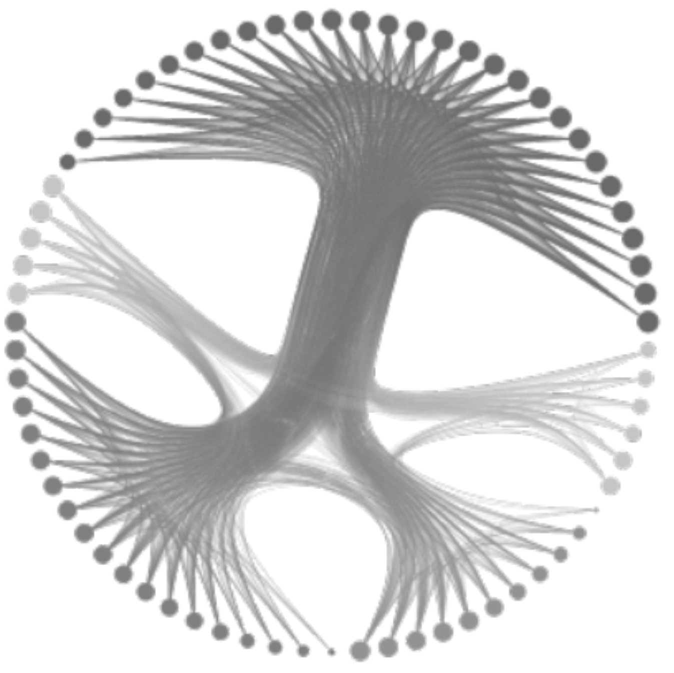
- Nodes j and k correlated through a common neighbor l
- **Shared-inputs** determine **stationary correlations**



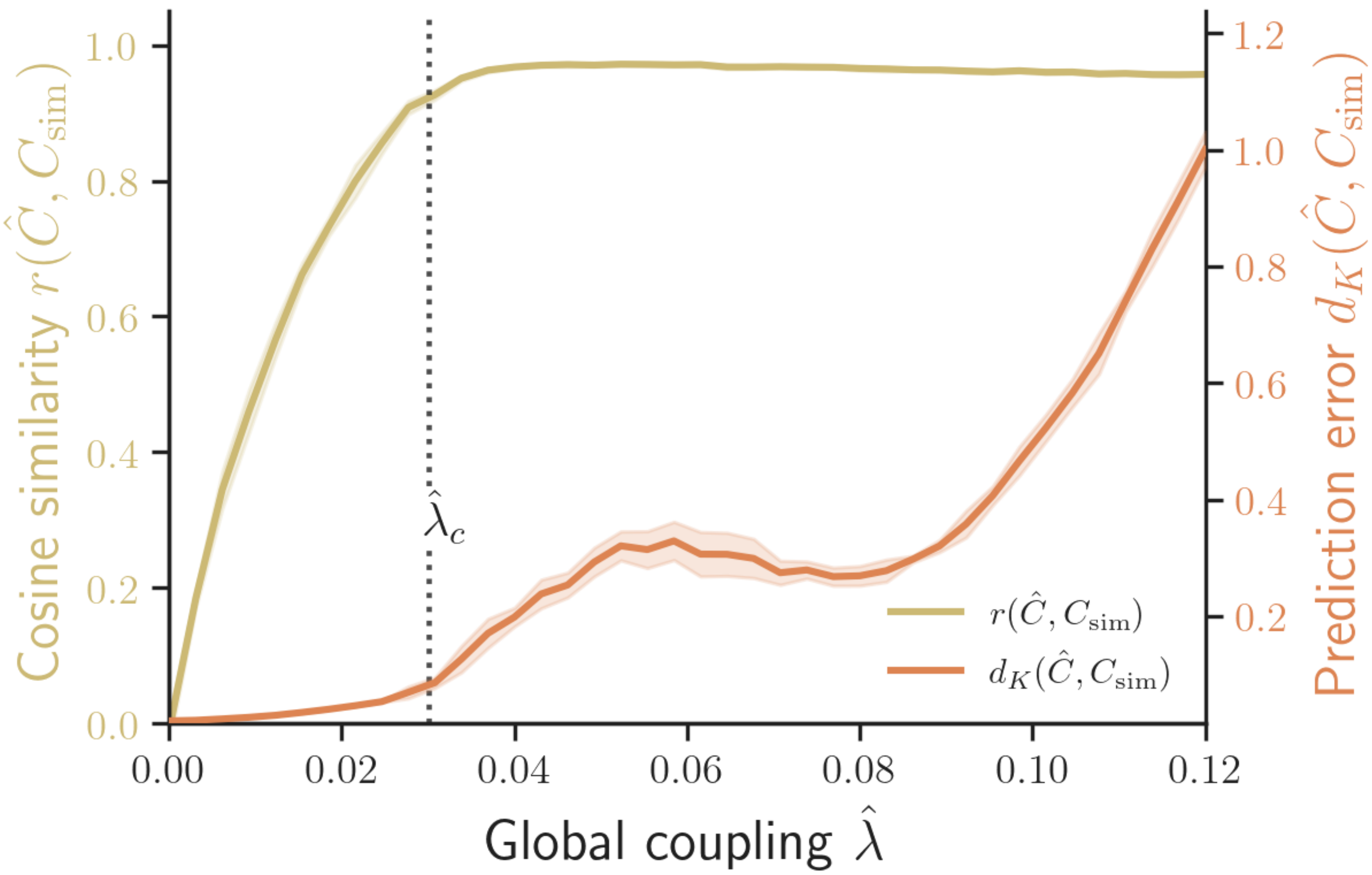
Validation on an empirical network



$$r(\hat{C}, C_{\mathrm{sim}})\big|_{\hat{\lambda}_c} = 0.991$$



Prediction accuracy as a function of the global coupling



Prediction error $d_K(\hat{C}, C_{\rm sim}) := \|\hat{C} - C_{\rm sim}\|_F / \|K\|_F$ exhibits a low plateau beyond the classical onset of global synchronization $\hat{\lambda}_c$ [4].

[4] J.G. Restrepo, et al. (2005), *Phys. Rev. E.*

Conclusion

There is **a closed-form mapping** $K \mapsto \hat{C}$ for functional connectivity, showing

Conclusion

There is **a closed-form mapping** $K \mapsto \hat{C}$ for functional connectivity, showing

(i) an absence of direct structural link in $\hat{C}$;
(ii) a linear dispersion–coupling tradeoff;
(iii) the crucial role of shared-inputs.

Conclusion

There is **a closed-form mapping** $K \mapsto \hat{C}$ for functional connectivity, showing

(i) an absence of direct structural link in $\hat{C}$;
(ii) a linear dispersion–coupling tradeoff;
(iii) the crucial role of shared-inputs.

Further analytical results

- Optimal global coupling for max. structure-function alignment
- Upper bound on structure-function alignment, solely dependent on K

Conclusion

There is **a closed-form mapping** $K \mapsto \hat{C}$ for functional connectivity, showing

(i) an absence of direct structural link in $\hat{C}$;
(ii) a linear dispersion–coupling tradeoff;
(iii) the crucial role of shared-inputs.

Further analytical results

- Optimal global coupling for max. structure-function alignment
- Upper bound on structure-function alignment, solely dependent on K

Future works

- Extension to the phase-lagged Kuramoto-Sakaguchi model

Conclusion

There is **a closed-form mapping** $K \mapsto \hat{C}$ for functional connectivity, showing

(i) an absence of direct structural link in $\hat{C}$;
(ii) a linear dispersion–coupling tradeoff;
(iii) the crucial role of shared-inputs.

Further analytical results

- Optimal global coupling for max. structure-function alignment
- Upper bound on structure-function alignment, solely dependent on K

Future works

- Extension to the phase-lagged Kuramoto-Sakaguchi model

Fonds de recherche Québec
UNIQUE CENTRE
CIMMUL
Centre interdisciplinaire en modélisation mathématique de l'Université Laval