The Control of Hamiltonian Chaos

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We demonstrate the control of the chaotic dynamics of Hamiltonian systems. This ability offers the possibility to select (stabilize) at will regular behavior(s) *within* the chaotic regime and to make efficient use of the richness and diversity of chaos.

I. INTRODUCTION

The counter-intuitive notion of chaos control is well summarized by the following statement due to Freeman Dyson [1]:

> A **chaotic motion** is generally neither predictable nor controllable. It is **unpredictable** because a small disturbance will produce exponentially growing perturbation of the motion. It is **uncontrollable** because small disturbances lead only to other chaotic motions and not to any stable and predictable alternative.

An operational definition of chaos is helpful to appreciate Dyson's assertion. In our presentation, *deterministic chaos* has a technical and precise meaning and despite a lack of a universal definition, most researchers would agree that it could be described as follows:
Chaos is a long-term aperiodic behavior of a dynamical system that possesses the property of sensitivity to initial conditions.
– long-term aperiodic behavior means that regularity (periodicity or quasi-periodicity) of the motion is absent.
– dynamical system indicates that determinism is present and that the source of the irregularity is inherent to that determinism and not to be found in a stochastic component.
– sensitivity to initial conditions implies that a very small deviation in the initial conditions is sufficient to create large deviations in the future states (the so-called "butterfly effect"), i.e. despite the presence of determinism, practical long-term predictability is lost.

This is the type of motion that Dyson had in mind. It is not new of course and it is clear that Maxwell and Boltzmann, the founders of statistical physics, were acutely aware of the property of sensitivity to initial conditions and its consequences. Not before Poincaré [2] could one ascertain the existence of this property in a system with *few* degrees of freedom, namely the reduced 3-body problem. It was not until 1990 however that Ott, Grebogi and Yorke (OGY) [3] addressed the question of control of chaos and described the theoretical steps necessary to achieve this goal. This method was very much in the spirit of von Neumann who imagined as early as 1950, that ' ' every unstable motion could be nudged into a stable motion by small pushes and pulls applied at the right places" [1]. The theoretical OGY work was rapidly followed by experimental verification [3]: von Neumann's dream had become reality.

This brief report describes some practical implementations for the recovery of order from chaos. Our examples are from the realm of conservative (Hamiltonian) systems. They are chosen because they have been much less studied than their dissipative counterparts, because their mixed (regular and chaotic) phase space offers new challenges to the standard control schemes and because of the growing evidence that the mere existence of Hamiltonian chaos [4] may shed new light on the foundations of statistical physics [5]. The stabilization of their chaotic behavior offers new grounds for a fascinating adventure. Many reviews on the control of chaos have appeared in the last few years and the reader may wish to consult the partial list given in [6].

II. A CONTROL STRATEGY

All stable processes, we shall predict.
All unstable processes, we shall control.
John von Neumann, circa 1950

In this Section, we show how the richness, the complexity and the sensitivity of chaotic dynamics can be used to select and stabilize at will, with small programmed perturbations, an otherwise unstable state of the natural dynamics. The goal is to achieve this feat *without* altering appreciably the original system. It is precisely the properties that differentiate a chaotic motion from an irregular or unstable behavior that are the solution to the control task. The important ingredients are:
– unstable periodic orbits (UPO) are typically dense in the chaotic attractor of dissipative systems or in the stochastic web of conservative systems, i.e. there are practically an infinity of unstable states to choose from.
– chaotic motion is ergodic, meaning that a chaotic trajectory will revisit infinitely often the neighborhood of any point within the available phase space.
– chaotic dynamics is sensitive to initial conditions, implying that *small* perturbations will naturally induce large effects.

To go beyond qualitative description, we establish some working conditions:

i. we suppose that the dynamics can be represented by a d-dimensional nonlinear map (either given explicitly or reconstructed from the observations)

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, p) \tag{1}$$

where the discrete time is labeled by n, and p is an accessible system parameter, the *control parameter*.

ii. there exists one or more specific UPOs for a given *nominal* value p_0 of the parameter, defined by

$$\{\mathbf{x}(i,p_0) : \mathbf{x}(i,p_0) = \mathbf{F}^{(m)}(\mathbf{x}(i,p_0),p_0)\ , \forall\ i = 1, m\} \tag{2}$$

for an orbit of period m, around which one wishes to stabilize the dynamics. Here $\mathbf{F}^{(m)}$ means the composition of $\mathbf{F}$ m times with itself.

iii. control is first initiated when a point, say $\mathbf{x}_N$, of the free trajectory falls in a small neighborhood of the UPOs, usually taken to be a ball $\mathcal{B}_\delta$ of radius δ around $\{\mathbf{x}(i,p_0)\}$,

$$||\mathbf{x}_N - \mathbf{x}(i,p_0)|| \leq \delta \qquad \text{for some} \quad i = 1, m \quad , \tag{3}$$

hereafter referred to as the *control* or $\delta-$*neighborhood*. Control is then kept active as long as the controlled trajectory stays within the prescribed $\mathcal{B}_\delta$ of $\{\mathbf{x}(i,p_0)\}$.

iv. we restrict the parameter variations δp, necessary to achieve control, to a maximum small perturbation

$$|\delta p| \leq |\delta p_{\max}| \ll |p_0| \tag{4}$$

defining the *control range*.

v. since the position of a periodic orbit is a function of p, and we assume that the local dynamics does not vary much within $|\delta p|$, a linear representation of the dynamics is possible.

For simplicity, we will confine our discussion to two dimensions. In 2D, the generic local neighborhood of a UPO is equipped with a stable and unstable manifold. A chaotic trajectory entering the neighborhood will move toward the UPO along the stable direction and escape along the unstable one. This is the "saddle dynamics" illustrated in Figure (1).

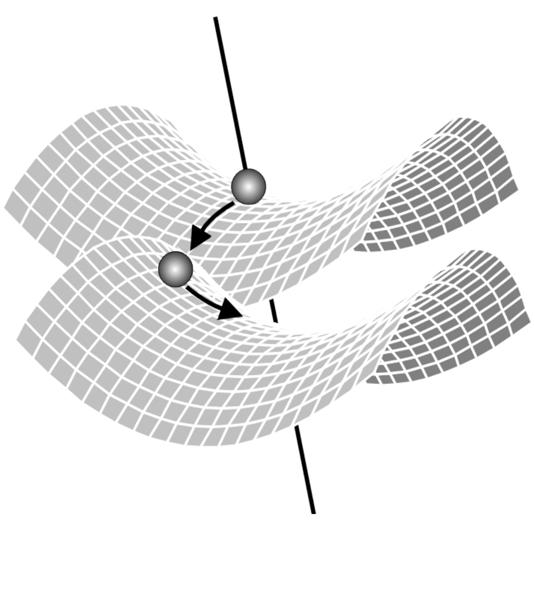

FIG. 1. **Saddle Dynamics**: Local geometry of control in 2D.

OGY [3] realized that a possible solution of controlling chaos could be obtained by locally displacing the manifolds in order to eliminate (at least to first order) the component of the motion along the unstable direction. The subsequent evolution would then naturally lead the orbit to the unstable point along the stable direction. This idea is geometrically presented in Figure (1) for an unstable fixed point, a situation much like the task of bringing a ball bearing to rest on a saddle. The remaining part of this section is devoted to the mathematical implementation of "small pushes and pulls applied at the right places".

In order to stabilize a chaotic trajectory around one of the existing UPOs, we have implemented a numerical version of the OGY method as modified in [7] for area-preserving mappings. This method is believed to be dynamically optimal in that it explicitly uses the local geometry of the underlying system. Given a dynamical system of the type (1), and a target UPO of period m, $\{\mathbf{x}(i,p_0)\}_{i=1,m}$, at some nominal parameter value p_0, one characterizes the local stable and unstable manifolds by the vectors $\mathbf{e}_{s,i}$ and $\mathbf{e}_{u,i}$ respectively as well as their contravariant counterparts $\mathbf{f}_{s,i}$ and $\mathbf{f}_{u,i}$ satisfying $\mathbf{f}_{u,i} \cdot \mathbf{e}_{u,i} = \mathbf{f}_{s,i} \cdot \mathbf{e}_{s,i} = 1$ and $\mathbf{f}_{u,i} \cdot \mathbf{e}_{s,i} = \mathbf{f}_{s,i} \cdot \mathbf{e}_{u,i} = 0$. The stabilizing perturbations $\delta p_n \equiv p_n - p_0$ are then obtained by firstly linearizing the dynamics in a δ-neighborhood of a member of the periodic orbit, say $\mathbf{x}(k,p_0)$, and around p_0, namely

$$\mathbf{x}_{n+1} - \mathbf{x}(k+1,p_n) \sim \mathbf{U}_k\ \left[\mathbf{x}_n - \mathbf{x}(k,p_n)\right] \tag{5}$$

where the $d \times d$ Jacobian matrix $\mathbf{U} \equiv D_{\mathbf{x}}\ \mathbf{F}(\mathbf{x},p)$ is evaluated at $[\mathbf{x} = \mathbf{x}(k,p_0),\ p = p_0]$ and it is understood that $||\mathbf{x}_n - \mathbf{x}(k,p_0)|| \leq \delta \ll 1$ and $|\delta p_n| \ll |p_0|$.

Secondly, the control criterion is imposed that $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n,p_n)$ should lie along the *stable direction* at $\mathbf{x}(k+1,p_0)$, i.e. $\mathbf{f}_{u,k+1} \cdot [\mathbf{x}_{n+1} - \mathbf{x}(k+1,p_0)] = 0$ which, together with the parametric variation of the periodic points, $\mathbf{x}(k,p_0+\delta p) \sim \mathbf{x}(k,p_0) + \mathbf{g}_k\ \delta p$, leads to the following expression for the parameter perturbation at the n-th iteration:

$$\delta p_n(\mathrm{OGY}) = -\ \frac{\mathbf{f}_{u,k+1} \cdot \{\mathbf{U}_k\ \left[\mathbf{x}_n - \mathbf{x}(k,p_0)\right]\}}{\mathbf{f}_{u,k+1} \cdot (\mathbf{g}_{k+1} - \mathbf{U}_k\ \mathbf{g}_k)} \quad . \tag{6}$$

Alternatively, one could do without the local manifolds by imposing instead that $||\mathbf{x}_{n+1} - \mathbf{x}(k+1,p_0)|| =$ minimum which results in a perturbation

$$\delta p_n(\mathrm{MED}) = -\ \frac{(\mathbf{g}_{k+1} - \mathbf{U}_k\ \mathbf{g}_k) \cdot \{\mathbf{U}_k\ \left[\mathbf{x}_n - \mathbf{x}(k,p_0)\right]\}}{||(\mathbf{g}_{k+1} - \mathbf{U}_k\ \mathbf{g}_k)||^2} \quad . \tag{7}$$

This modification was first introduced in [8] and goes by the name of minimal expected deviation (MED) method.

The two schemes give similar performances, although (6) is somewhat more demanding numerically.

In summary, the stabilization procedure can be divided in three separate stages: the *learning stage*, where one identifies the desired UPOs, extracts the Jacobian matrices, and (for the OGY scheme) calculates the corresponding stable and unstable directions $\mathbf{e}_{s,i}$, $\mathbf{e}_{u,i}$ to construct the contravariant vectors $\mathbf{f}_{u,i}$; the *transient stage*, where, after randomly choosing an initial condition, the system is let to evolve freely at the nominal parameter value p_0 until, at the *control stage*, once the chaotic trajectory has entered the prescribed δ-neighborhood, the control is attempted by means of small parameter perturbations.

III. CHAOTIC DYNAMICS UNDER CONTROL

We have selected 2 examples for their novelty and complexity. ALL the relevant control informations are obtained numerically. Reliable methods (see e.g. [9]) exist to locate the positions of the UPOs and we will assume hereafter that their locations are known prior to the control session. The numerical construction of the Jacobian matrices from time series is often a subtle task and is beyond the scope of this article. The reader is referred to [10] for technical details. In the following, the value of the control parameter around which small perturbations are applied is denoted by a sub-index 0.

A. Billiards: Chaos coming to Light

The study of the frictionless motion of a particle bounded by a closed surface where it is specularly reflected is known as *billiard dynamics* and dates back to Birkhoff [11]. It serves to illustrate the transition from strict regularity (integrability) to chaos (ergodicity) in Hamiltonian systems [11] and bears important connections to *quantum chaos* as well [12]. We have chosen to study the 2D *cosine billiard* where the surface is parameterized in polar coordinates by the relation

$$r(\phi) = 1 + \epsilon \cos\phi \quad . \tag{8}$$

Geometrically, the parameter ϵ is a measure of the asymmetry of the surface with respect to circularity, and dynamically, it is a measure of nonintegrability since $\epsilon = 0$ (the circle) represents the integrable limit. For all $\epsilon \neq 0$, there are finite regions of phase space that contain chaotic trajectories. Figure (2) shows on the left the mixed and complex structure of phase space for $\epsilon_0 = 0.3$: the state variables are the incident angles on the surface, $\{\alpha_n\}$, and the polar angles of the point of impact, $\{\phi_n\}$. Since motion is free in between collisions with the surface, our example belongs to the class of 2D area-preserving mappings, where attractors are absent and replaced by stochastic bands mixed with regular regions. Within these bands, the motion is ergodic: the blackened region is produced by *one* single chaotic orbit. Embedded in this stochastic web, one observes a number of UPOs whose physical trajectories inside the boundary are shown in the middle portion of Figure (2). By pulsating the deformation parameter ϵ about its nominal value ϵ_0, we have achieved control of 3 UPOs of period 4, 5 and 9. We used the MED control algorithm with a neighborhood of $\delta = 10^{-2}$. A numerical OGY method gives identical performance.

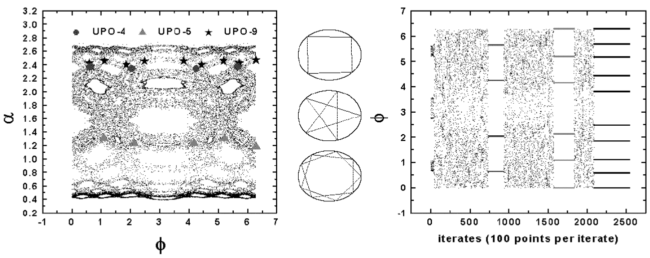

FIG. 2. **Cosine Billiard**: (*left*) mixed chaotic (filled) and regular (open islands) phase space with embedded UPOs for $\epsilon_0 = 0.3$; (*middle*) MED controlled UPOs of period 4, 5 , 9; (*right*) stabilized ϕ variable of the corresponding UPOs held for 5 000 cycles each with $\delta = 10^{-2}$ (1 cycle = 1 complete orbit).

We mention that the successful control of billiard dynamics may offer a solution to the degradation of finesse in resonant optical microcavities [13]. It has been inferred that the loss of lasing activities might be associated with ray chaos (geometrical limit) in the optical resonators where the photons are transported (via chaotic diffusion) to regions of phase space where refractive escape (Snell's law) becomes possible. The (generally non-spherical) dielectric droplets making up the resonators behave very much like 2D billiards and we propose that programmed variations of their asymmetry may help reduce photon leakage. The viability of the proposal is currently being investigated. Further, the most powerful microlaser in existence [14] has a mostly chaotic phase space (very similar to the one studied here) and (classical) nonlinear dynamics offers the explanation for its high-power, high-directional emission. Our ability to control its dynamics would allow us to change at will the emission patterns and wavelengths. Chaos has come to light!

B. Flows: DKP revisited

Our next example is a continuous, 2 degrees of freedom (4D phase space) Hamiltonian system. It represents the motion of an electron under the combined influence of a Coulomb and a magnetic field. It goes under the name, *diamagnetic Kepler problem* (DKP), and occupies central stage in classical and quantum chaos research [12]. We use atomic units and semi-parabolic coordinates ($\mu^2 = r + z$, $\nu^2 = r - z$, $t = 2r\tau$) with generalized momenta $p_{\mu,\nu} = d\mu, \nu/d\tau$ to write a resulting *scaled* Hamiltonian (for angular momentum $L = 0$)

$$(\mu^2 + \nu^2)\gamma^{-2/3} H_{DK} = \frac{1}{2}(p_\mu^2 + p_\nu^2) - 2 + \frac{1}{8}\mu^2\nu^2(\mu^2 + \nu^2) = (\mu^2 + \nu^2)\ \epsilon \quad , \tag{9}$$

where the Coulomb singularity has been explicitly removed. The scaled energy ϵ is related to the physical energy E by $\epsilon = \gamma^{-2/3}\ E$ where $\gamma = B/B_c$ denotes the strength of the magnetic field relative to the unit $B_c \simeq 2.35\,10^5\ T$. It has proven useful to consider, instead of (9), the Hamiltonian function with a fixed (pseudo)-energy equal to 2, namely

$$\hat{h}_{DK} = \frac{1}{2}(p_\mu^2 + p_\nu^2) - \epsilon\,(\mu^2 + \nu^2) + \frac{1}{8}\mu^2\nu^2(\mu^2 + \nu^2) \equiv 2\ , \tag{10}$$

where now ϵ appears as a dynamical parameter. As ϵ is varied, the classical flow of (10) covers a wide range of Hamiltonian dynamics reaching from bound, nearly integrable behavior to completely chaotic and unbound motion [15].

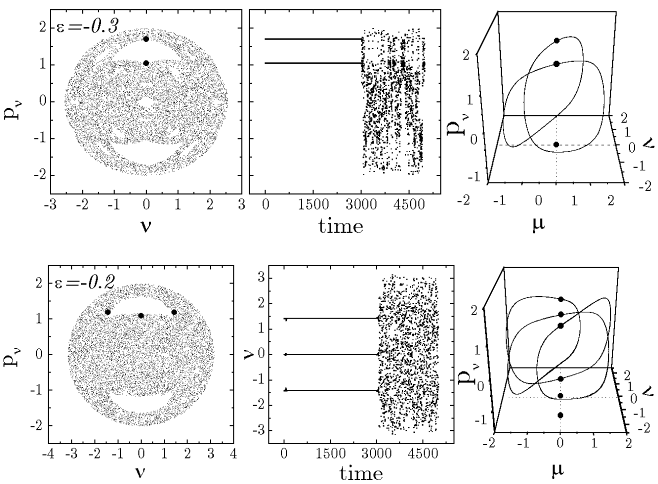

FIG. 3. **Diamagnetic Kepler Problem** for 2 scaled energies $\epsilon_0 = -0.3$ (period 2) and $\epsilon_0 = -0.2$ (period 3) in top, and bottom panels (respectively): (*left*) Poincaré section (PS) $\mu = 0, \dot{\mu} > 0$ showing one chaotic trajectory (filled space) and the OGY controlled UPO (black dots); (*middle*) the stabilized p_ν or ν variable for the first 3000 intersections with the PS before control is turned off; (*right*) corresponding 3D stabilized trajectory.

The dimension reduction (from 4D to 2D) and discretization is performed by observing the dynamics on the Poincaré section defined by $\mu = 0, \dot{\mu} > 0$. The energy shell is then mapped to an area bounded by the condition $p_\nu^2 - 2\epsilon\ \nu^2 = 4$ which represents an ellipse in the (ν, p_ν) plane. The left panels of Figure (3) shows the collection of points $\{\nu_n, p_{\nu,n}\}$ obtained by numerical integration of the equations of motion for $\epsilon_0 = -0.3$ and -0.2. One notices, for these energies, that phase space has few regular structures: apart from few lobes of regularity, the rest of the ellipse is filled by the successive piercings of *one* chaotic trajectory. The black dots indicate the positions of the UPOs. We have succeeded in stabilizing a number of UPOs for the system, two of them (periods 2 and 3 for $\epsilon_0 = -0.3$ and -0.2 respectively) are displayed with their 3D trajectories in Figure (3).

In attempting to bring order to the DKP dynamics, we had to overcome a number of difficulties not encountered in previous studies. First, a typical trajectory spends a lot of time away from the Poincaré section and because of the sensitivity of the dynamics we had to device an efficient variable step *symplectic* integrator [16] thereby preserving the geometrical structure of the Hamiltonian. Second, we had to obtain numerical Jacobian matrices (solely with informations gathered on the Poincaré section) for all members of the UPOs because it was found to be necessary to intervene at *every* crossing of the Poincaré section. Third, the eigenvalues of area-preserving Jacobian matrices are often complex and the stable and unstable manifolds are no longer along the directions of their eigenvectors. A new method had to be implemented. Details of the solutions to these problems can be found in [17].

We should point out that this is the first control of a realistic Hamiltonian system and the first complete numerical (integration, determination of the Jacobian matrices and the local manifolds) implementation of the OGY strategy. It still remains an open question however if manipulations of the magnetic field to induce stabilization of a classical unstable orbit can be extended to the realm of semi-classical physics. For example, certain classes of non-spreading Rydberg wave packets localized around *classical* circular or elliptical Kepler trajectories [18] appear as prime candidates for future investigations.

IV. PROPERTIES OF THE CONTROL SCHEME

The lessons learned through the previous examples and many more not reported here allow us to draw a list of the properties and advantages of the adopted control 'philosophy' and to point to remaining difficulties.

– no model dynamics is required *a priori* and only *local information* is needed;
– computations at each step are minimal;
– *gentle touch*: the required changes in p_0 can be quite small ($< 1\%$);
– *multi-purpose flexibility:* different periodic orbits can be stabilized for the *same* system in the *same* parameter

range;
– control can be achieved even with imprecise measurements of eigenvalues and eigenvectors: the methods are robust;
– the methods can also be applied to *synchronization* of several chaotic systems.

At least three complications come to mind when one considers the implementation of chaos control strategies to the laboratory. The presence of noise, ignored so far, may induce occasional loss of control or hinder it altogether. The average waiting time to fall in the δ-neighborhood may be very (too) long, especially for Hamiltonian systems and a *targeting* strategy [19] should complement the control method. Furthermore, the system's parameters may drift with time and this nonstationarity should be accounted for by updating the control informations. *Tracking* [20] is the name given to this procedure.

V. CONCLUSIONS AND FUTURE CHALLENGES

We have presented some of our efforts to recover order from chaos in Hamiltonian systems and we have discussed some of the basic techniques for controlling their chaotic motion. These same techniques (with slight modifications) apply equally well to dissipative systems. Applications of the control of chaos have been reported in such diverse areas as aerodynamics, chemical engineering, communications, electronics, fluid mechanics, laser physics, as well as, biology, finance (not confirmed!), medicine, physiology, epidemiology and the list is constantly growing. Some of the earliest *experimental* successes of the methods can be found in [21].

Although the last decade has seen much accomplishments, challenges for the future are still numerous: generalization to spatio-temporal chaos, adaptive control for non-stationary dynamics, effective control in the presence of noise (dynamical and/or observational), adaptive synchronization of chaos are some of the things that have not yet reached maturity.

However, the greatest challenge will remain for some times the application to complex biological systems and in particular to brain dynamics [22]. Complex natural systems are noisy, contain a strong stochastic component and are not endowed with a behavior called chaos (at least not in its mathematical rigorous sense). Yet, one would like to believe that "the controlled chaos of the brain is more than an accidental by-product of the brain complexity" [23]. The perspective of unifying the techniques of deterministic chaos control with a statistical stochastic description as a possible therapeutic strategy against dynamical diseases is surely something to consider.

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The first theoretical and experimental implementation of "gentle" *chaos control* appears in [3]. We have collected in [6] some of the most *recent reviews on controlling chaos*. The December issue (1997) of the journal *Chaos* contains a wealth of useful information. It is safely the place to start to deepen one's understanding of the manipulation and efficient use of chaos. Questions of integrability and chaos in *Hamiltonian systems* are superbly treated in [4] and the implications of the existence of Hamiltonian chaos are discussed in [5]. The basics of *billiard dynamics* can be acquired in [11] and the fascinating concept of *quantum chaos* is elegantly presented in [12].

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