Geometric evolution of complex networks

Dynamica research group

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Summary

Real complex networks feature scale-free degree distribution $P(k) \sim k^{-\gamma}$, high clustering coefficient and assortativity (dissortativity). Whereas the first two properties are well reproduced by some network growth mechanisms, such as **preferential attachment** (PA) [1] and **network geometry** [2], it is fair to say that assortative mixing patterns have, for the most part, escaped a systematic description.

- We present a growing geometric networks model where new nodes connect **homogeneously** with the existing nodes.
- We calculate the model's degree distribution, degree correlation and clustering coefficient via a hidden variable framework.
- The parameters of the model can be tuned to reproduce any **degree distributions** and **assortative mixing patterns**.

Model

Growth process

Consider the surface of a $(\mathcal{D}+1)$-dimensional sphere, noted $\mathbb{S}^{\mathcal{D}}$, as the embedding space of the geometric networks generated. The growth process goes as follow:

1) At time $t \geq 1$, a new node (noted t) is assigned a random position x_t uniformly distributed on $\mathbb{S}^{\mathcal{D}}$.
2) Node t connects with the existing nodes $s < t$ with *time-varying* connection probability $p[x_t, x_s; \mu(t), \beta]$.
3) Steps 1 and 2 are repeated until a total of N nodes has been reached.

Fermi-Dirac connection probability

We choose the following **Fermi-Dirac** connection probability:

$$p[x, y; \mu(t), \beta] = \frac{1}{\exp\{\beta[d(x, y) - \mu(t)]\} + 1}$$

- $\mu(t)$: chemical potential
- β: inverse temperature
- $d(x, y)$: distance between x and y on $\mathbb{S}^{\mathcal{D}}$.

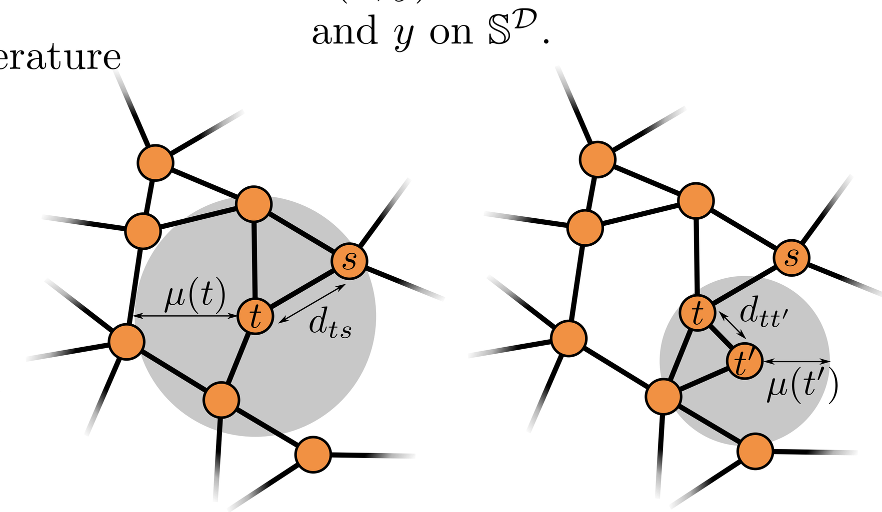


Illustration of the growth process on $\mathbb{S}^2$: $\beta \to \infty$, $d_{ts} \equiv d(x_t, x_s)$. The gray area corresponds to a region centered on x where $p[x, y; \mu(t), \beta]$ is non zero.

Analytical results

With the hidden variable framework of [3], where the variables are $h = (x, t)$, a **detailed analysis** of the model's structural properties becomes accessible.

Degree sequence

We calculate the expected degree $\kappa(\tau)$ of each node with respect to its normalized birth time $\tau \equiv t/N$:

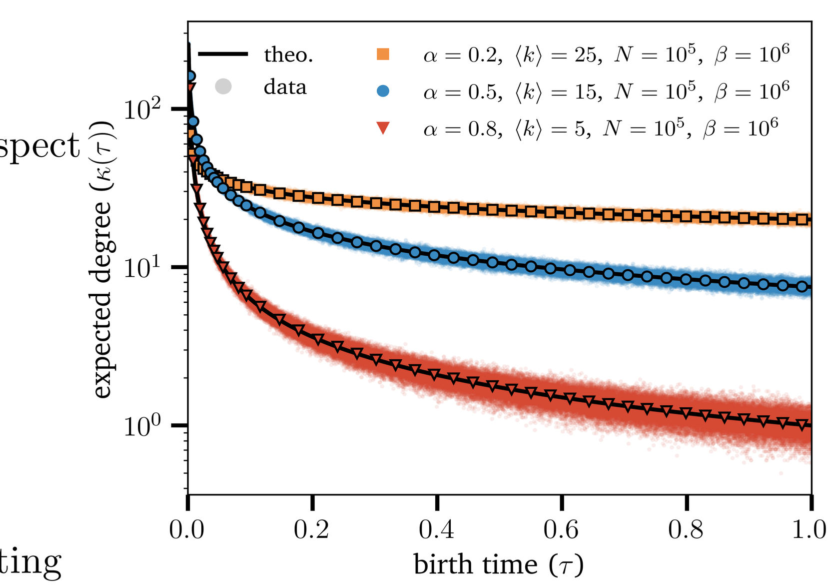
$$\kappa(\tau) = N\left[\tau n(\tau) + \int_\tau^1 n(\tau')d\tau'\right]$$

where

$$n(\tau) = \int_{\mathbb{S}^{\mathcal{D}}} p[x_t, y; \mu(\tau N), \beta]\rho \, dy$$

is the averaged probability that node t connects to any existing nodes, with ρ as the density of nodes on $\mathbb{S}^{\mathcal{D}}$. Node t effectively connects **homogeneously** with probability $n(\tau)$ with any existing nodes.

- The chemical potential $\mu(t)$ can be chosen to reproduce **any degree sequences**.



Expected degree of each node τ for scale-free networks ($\kappa(\tau) \propto \tau^{-\alpha}$): $N = 10^5$, $\gamma = 1 + 1/\alpha$. Results averaged over 96 instances.

Degree correlation

We use the average degree of the nearest neighbors (ANND) $\kappa_{nn}(\tau)$ to characterize the degree correlation given by

$$\kappa_{nn}(\tau) = N\left[\int_0^\tau \frac{\kappa(\tau')n(\tau)}{\kappa(\tau)}d\tau' + \int_\tau^1 \frac{\kappa(\tau')n(\tau')}{\kappa(\tau)}d\tau'\right]$$

- The **ordering** in which nodes appear in the network affects the degree correlation.

Clustering coefficient

We find an integral expression of the local clustering coefficient $c(\tau)$ of node t and investigate two limit cases.

- **Geometric phase** ($\beta \to \infty$): $\langle c \rangle = \mathcal{O}(1)$;
- **Random phase** ($\beta \to 0$): $\langle c \rangle = \mathcal{O}(N^{-1})$.

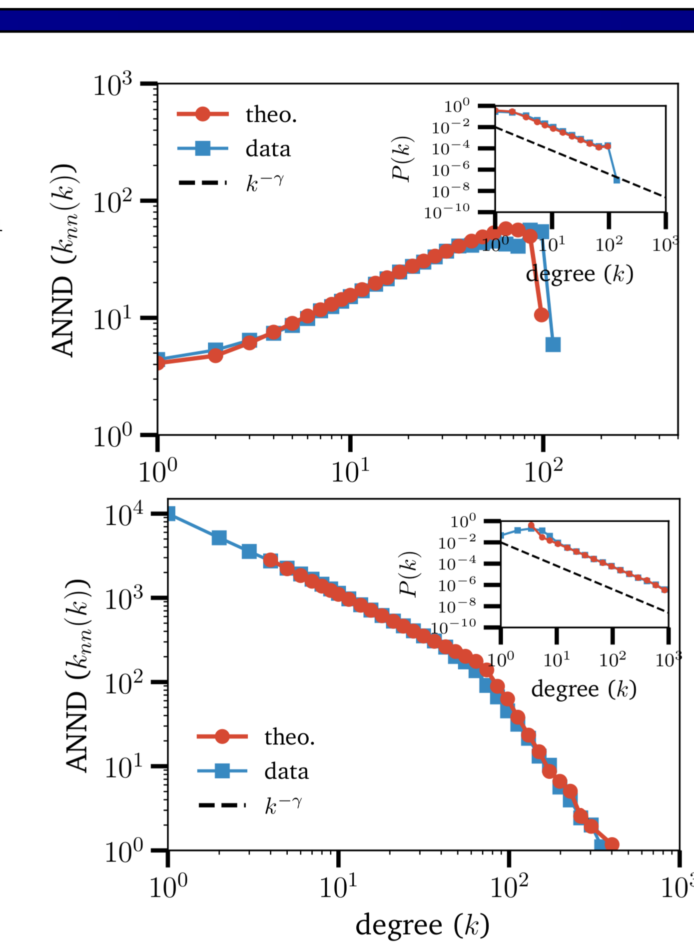
Network History

Let us consider a generated network with a specific degree sequence $\{k_i\}$. We define the **network history** as

$$H = \{\tau_i\}, \text{ such that } \kappa(\tau_i) = k_i$$

where $\tau_i \in [0, 1)$.

- Although different histories H, with an appropriate choice of $\mu(t)$, can be made to respect $\{k_i\}$, they lead to different structural properties;
- The selection of H can accommodate our need for **assortative** or **disassortative** behaviour.

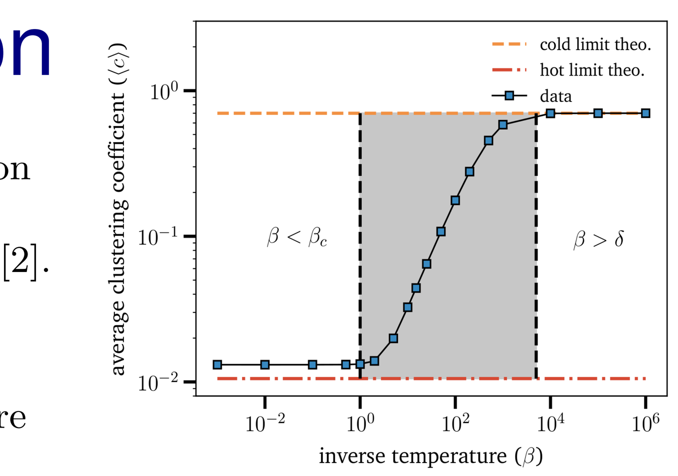


Degree correlation (ANND) for scale-free networks with different histories.
(top) An decreasing degree ordered history (**assortative**)—the high degree nodes are old while the low degree ones are young ($\kappa(\tau) \propto \tau^{-\alpha}$).
(bottom) An increasing degree ordered history (**disassortative**)—the high degree nodes are young while the low degree ones are old ($\kappa(\tau) \propto (1-\tau)^{-\alpha}$).
We used $N = 10^4$, $\alpha = 0.83$ ($\gamma = 2.2$), $\langle k \rangle = 6$ and $\beta = 10^6$. Results averaged over 48 instances.

Phase transition

A phase transition occurs in the clustering coefficient as a function of the inverse temperature β. A similar result was found in Ref. [2]. We estimate the **critical threshold** to $\beta_c \sim 1$.

- When $\beta < \beta_c$, the networks are **random**;
- When $\beta > \beta_c$, the networks are **geometric**;
- The phase transition is present for **any degree sequences**.

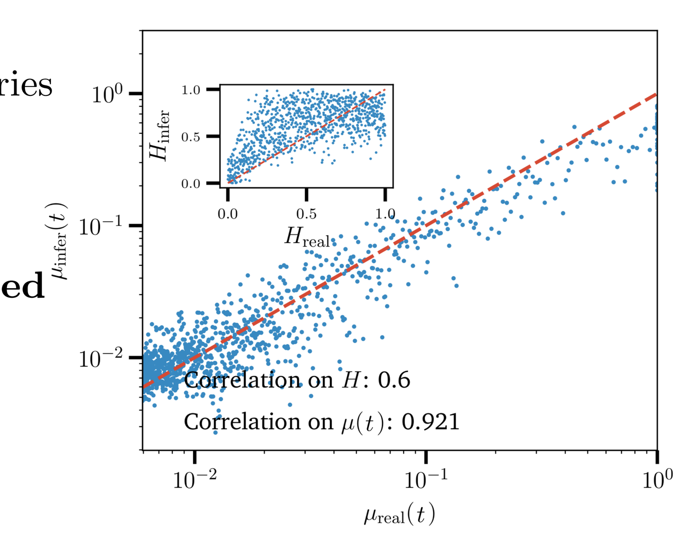


Phase transition of the clustering coefficient for scale-free networks ($\kappa(\tau) \propto \tau^{-\alpha}$): $N = 10^4$, $\alpha = 0.5$ ($\gamma = 3$), $\langle k \rangle = 50$. The dashed lines correspond to analytical solutions for the cold limit ($\beta \to \infty$) and the hot limit ($\beta \to 0$), and $\delta \sim \frac{N}{2}$ corresponds to the clustering saturation point. Results averaged over 20 instances.

Preliminary results: Inference

Using a MCMC algorithm, we sample scale-free networks histories and reinfer the parameters $\mu(t)$ with only the **structure as a prior information**.

- $\mu_{\text{infer}}(t)$ is **strongly correlated** with $\mu_{\text{real}}(t)$;
- A similar procedure could be used on **real complex networks**;
- Geometric evolution is an **effective growth process**.



Inference of $\mu(t)$ for synthetic scale-free geometric networks ($\kappa(\tau) \propto \tau^{-\alpha}$): $N = 10^3$, $\alpha = 0.83$ ($\gamma = 2.2$), $\langle k \rangle = 35$, $\beta = 10^6$. The inset shows the inference of the network history.

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Bibliography

This work is presented in detail in:

C. Murphy *et al.*, **arXiv:1710.01600 (2017)**.

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