Complex networks are an emerging property of hierarchical preferential attachment

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Summary

Scale independence is observed in all aspects of human life and often modeled through preferential attachment (PA). Network science and PA processes tend to focus on one feature at a time; e.g. degree distribution [1] or community structure [2].

Complex networks are constructs obtained by projecting complex hierarchical systems on a set of nodes and links; collapsing geographical/age/cultural/professional correlations.

Why not directly model the hierarchical system itself instead of its projection?
What can emerge from a simple hierarchy of scale independent organizations?

Hierarchical Preferential Attachment features

- the simplicity of preferential attachment,
- complex networks as an emerging property.

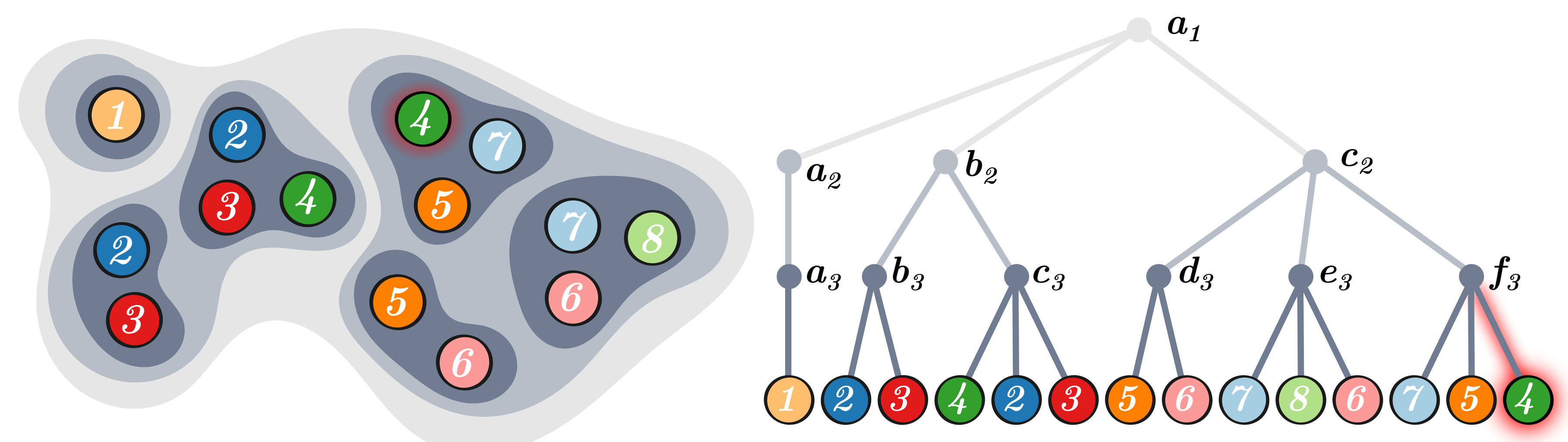
Complex networks emerge from hierarchy?

Hierarchical systems produce networks when projecting under a chosen level of structure. Correlations inter and intra levels of structures dictate properties of the network:

- locally: degree and clustering;
- globally: centrality, self-similarity;
- + complex properties such as geometrical mapping!

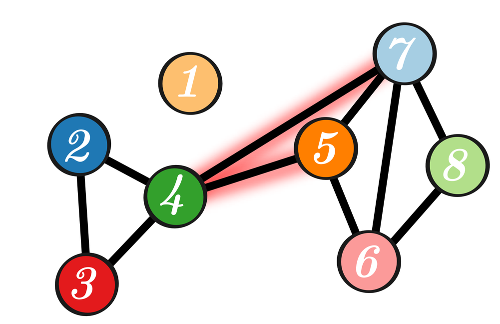
Hierarchy makes complex networks complex.
HPA is perfectly suited to model scale-independent networks.

Hierarchical Preferential Attachment (HPA)



HPA ≡ colored balls are thrown in embedded bins.

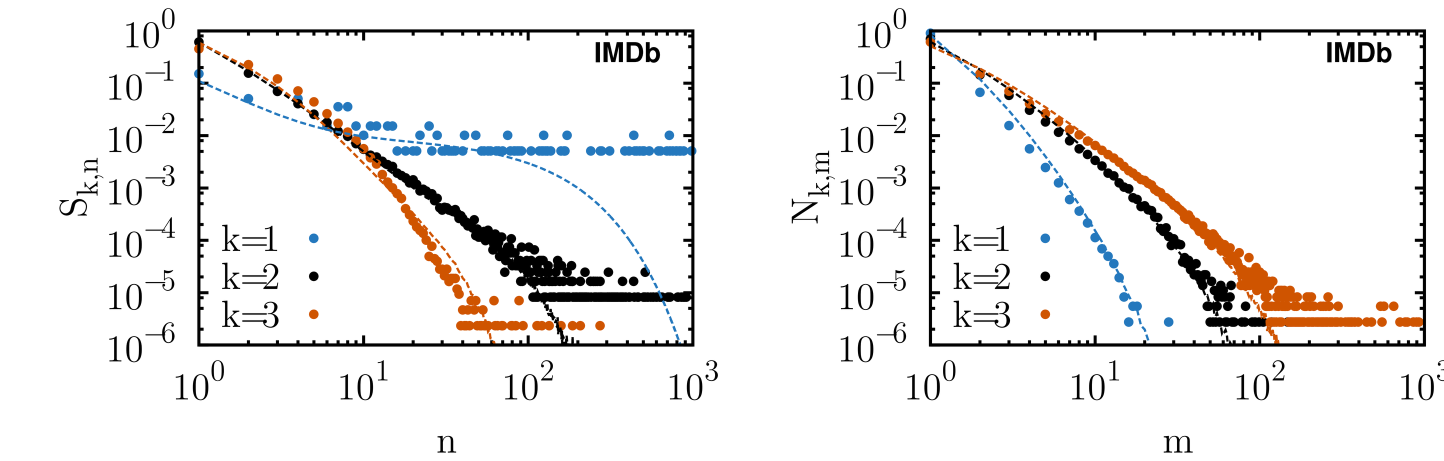
- Embedded bins (top left) represent a tree-like hierarchy (top right).
- Balls/bins are different structural levels (e.g. people, communities, cities, countries).
- At level i, let $p_i \equiv$ probability that a ball falls in a new bin;
  $q_i \equiv$ probability that the color of the ball is new for that bin.
- Whenever an existing bin and/or color has to be chosen, it is done preferentially to its size/frequency at that level.
- A network is obtained by projecting the system on a given level.
- For the network on the right: colors found in a common bin on the lowest levels are linked in the network. This could be a network of collaborating scientists, projecting labs (level 3) across cities (level 2) and countries (level 1) on a single social network.
- Other projections are possible; e.g., a network of the boxes of level 3 that share at least one color could be a network of collaborations between research groups.



Case study: movie production structure

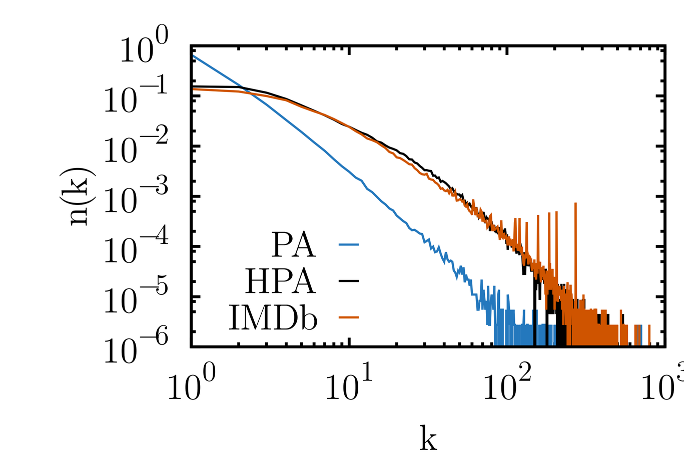
Hierarchy: **countries** (largest bins, level $k=1$) containing **production companies** (middle bins, level $k=2$) producing **movies** (smallest bins, level $k=3$) with **producers** (colored balls).

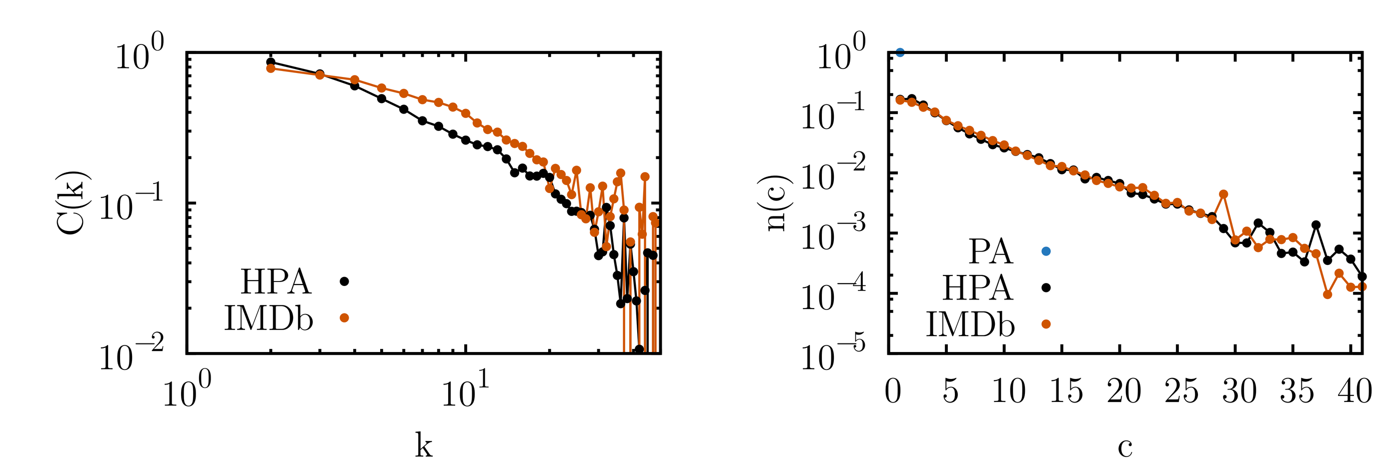
We set all $\{p_i, q_i\}$ with $S_{k,n}$ (distribution of level k structures of sizes n) and $N_{k,m}$ (distribution of colors appearing in m level k structures) by comparing data (dots) and simulations (lines).



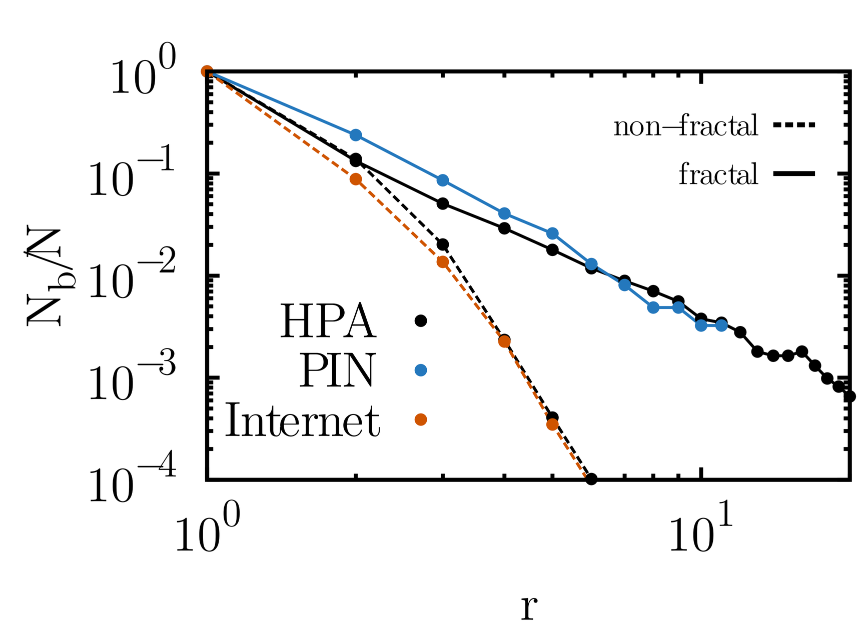
Projection for a realization of HPA:

- Project the system in a network of co-producing credits: links between producers who have produced together, regardless of companies and country.
- Random **HPA network** captures structure from real network not captured by Standard PA:
  1. degree distribution $n(k)$
  2. local clustering coefficient $C(k)$ around nodes of degree k ($C(k) = 0 \; \forall \; k$ in Standard PA)
  3. distribution $n(c)$ of coreness c, i.e. number of nodes in a shell of the k-core decomposition ($n(c) = \delta_{c,1}$ in Standard PA)





Proof of concept: Fractality and geometrical mapping



Fractal (& non-fractal) networks from hierarchy:

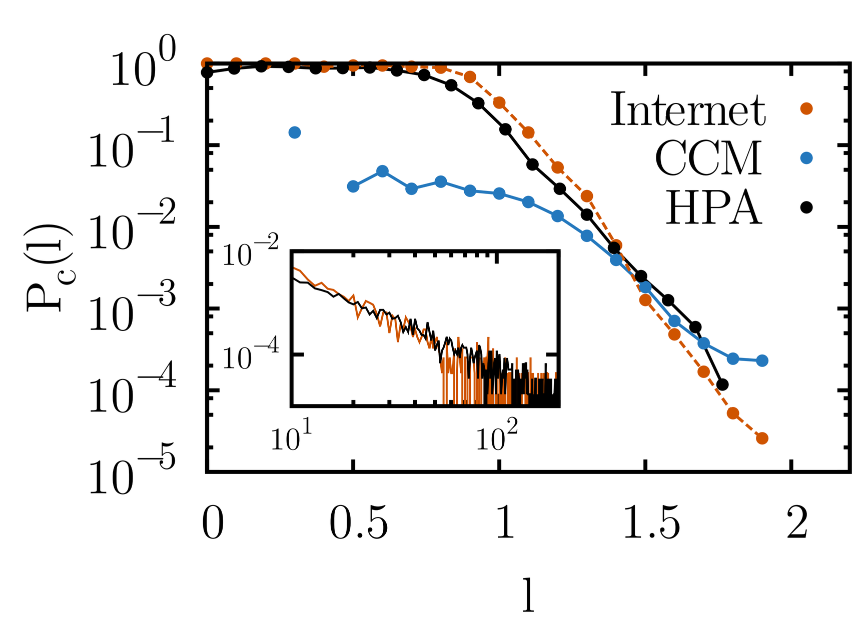
HPA yields fractal and non-fractal networks: self-similarity might imply hierarchy, the opposite is not true.

- Well-mixed hierarchies have a network diameter D scaling with the logarithm of the number of nodes N (non-fractal)
- Systems with well defined hierarchy lead to a power-law relation between D and N (fractal)

Fractality is uncovered with box-counting [3]: groups of nodes within a distance r (number of links) are assigned to the same box. The fractal dimension d_b relates the number N_b of boxes and their size r: $N_b \propto r^{-d_b}$.

Figure on the left: box counting results on a fractal network (protein interaction network of Homo Sapiens) and a non-fractal network (the Internet at the level of autonomous systems) [3].

∴ **HPA models how both of these networks span and cover their respective space.**



Hyperbolic mapping of networks [4]:

Mapping of a network: assign geometrical positions to nodes to embed the network in an hyperbolic space. Nodes close (in links) in the network must be geometrically close (in space).

Navigability of complex networks:

- predicts existence of links as a function of geometrical distance between nodes, enabling an efficient navigation.
- is not captured by classical preferential attachment.

Figure on the left: probability of connection $P_c(l)$ between nodes at a distance l after an inferred projection of the networks unto an hyperbolic space [4].

- The Internet and its HPA model share a similar scaling exponent for their degree distribution (inset).
- The CCM (Correlated Configuration Model) corresponds to a rewired Internet preserving degree distribution and degree-degree correlations, but obviously lacking the more complex structural correlations.

∴ **Geometrical constraints can emerge simply from hierarchy.**

Bibliography and Acknowledgements

HPA is presented in: L. Hébert-Dufresne *et al.*, arXiv:1312.0171 (2013)
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