Functional connectivity in neural oscillator networks analytically predicted from common-neighbour structure

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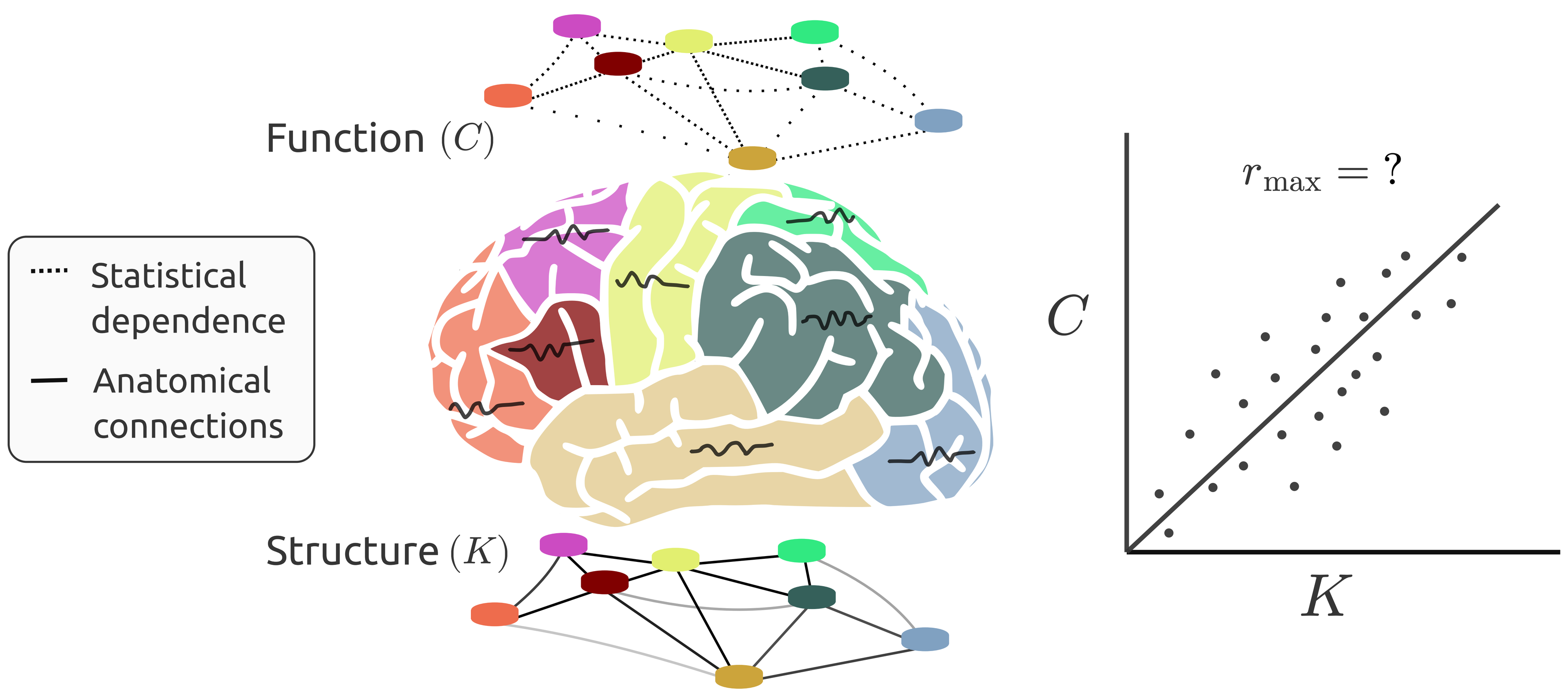
Structure-function relationship

Functional connectivity (FC) describes statistical dependencies between the activity of neurons or groups of neurons [1]. Comparing FC with anatomical or structural connectivity (SC) has emerged as a promising avenue to study how brain structure supports function and how both change in disease or with cognition [2]. However, empirical studies across species and recording modalities have reported a wide range of FC-SC correspondence values, typically assessed using Pearson correlation [1,3]. Recent theoretical work further suggests that fundamental limits on the information shared among structure and dynamics may constrain our ability to relate FC to SC [4,5].

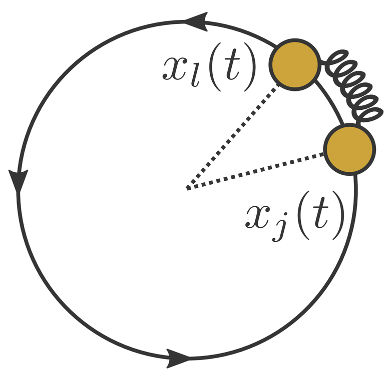


Interpreting structure-function relationships thus remains challenging and raises key questions: *what is the strongest FC-SC correspondence*, and *is there an optimal regime associated with such relationship?*

Goal: predict analytically the optimal SC-FC regime

This project aims to identify the dynamical regime in which the structure-function correspondence is maximized. To assess how structural connectivity shapes functional dependencies, we derived closed-form relationships between neural activity correlations and the underlying SC. A computational framework (see SIMBA library) was developed to allow for the numerical validation of our analytical predictions.

Method: perturbative approach on neural oscillators



Kuramoto model of coupled oscillators

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \omega_j + \frac{\lambda}{N}\sum_{l=1}^{N} K_{jl}\sin(x_l - x_j)$$

- x_j : activity of neuron j
- ω_j : natural frequency j
- λ : global coupling
- K : connectivity matrix

(a) Weak-coupling perturbative expansion around the uncoupled state ($\lambda = 0$)

$x_j(t) \approx x_j(0) + \omega_j t + \frac{\lambda}{N}u_j^{(1)}(t) + \frac{\lambda^2}{N^2}u_j^{(2)}(t)$ ⟶ coactivity: $c_{jk}(t) = \sin x_j \sin x_k$

$c_{jk}(t) \approx c_{jk}^{(0)}(t) + \frac{\lambda}{N}c_{jk}^{(1)}(t) + \frac{\lambda^2}{N^2}c_{jk}^{(2)}(t)$ ⟶ time-correlation: $C_{jk}(T) = \langle 2c_{jk}\rangle_T$

(b) Time-average and ensemble-average via integration over natural frequencies

$\langle\cdot\rangle_T = \frac{1}{T}\int \ldots \,\mathrm{d}t$, $t \in [0, T]$

$\langle\cdot\rangle_\omega = \int (\cdot)\times \rho(\omega)\,\mathrm{d}\omega$

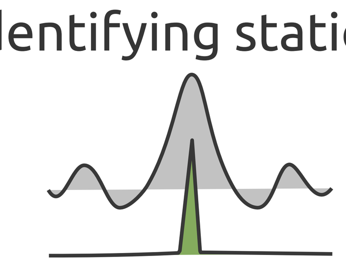
(c) Resonance condition as a filter for identifying stationary correlations

i) Ensemble-average

$$\hat{C}_{jk}(T) = \langle C_{jk}(T)\rangle_{\boldsymbol{\omega}}$$

ii) Resonance

$$\sum_{\ell} n_\ell \omega_\ell = 0$$



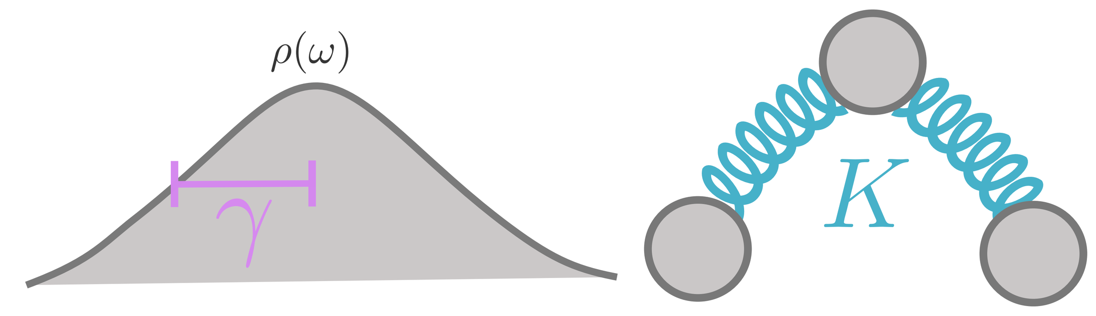
iii) FC prediction

$$\hat{C}_{jk} := \lim_{T\to\infty}\hat{C}_{jk}(T)$$

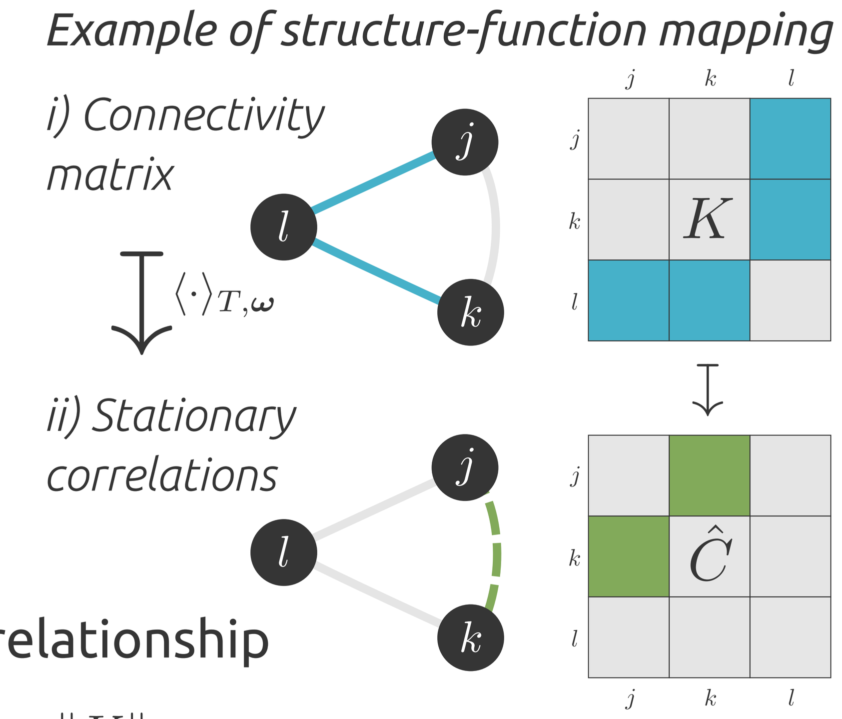
Analytical predictions

(a) Stationary correlation matrix (FC prediction)

$$\hat{C}(\gamma, \lambda, K) = \mathrm{I} + \frac{5\lambda^2}{4\gamma^2 N^2}\left(KK^\top - \mathrm{diag}(KK^\top)\right)$$



Example of structure-function mapping



(b) Reconstruction error as a measure of SC-FC relationship

$$d(\lambda) = \|\underbrace{(\hat{C}(\lambda) - \mathrm{I})}_{\text{"Function"}} - \underbrace{K}_{\text{"Structure"}}\|_F \;/\; \underbrace{\|K\|_F}_{\text{Normalization}}$$

(c) Analytical optimization of SC-FC reconstruction error

i) Optimal coupling of strongest SC-FC relationship

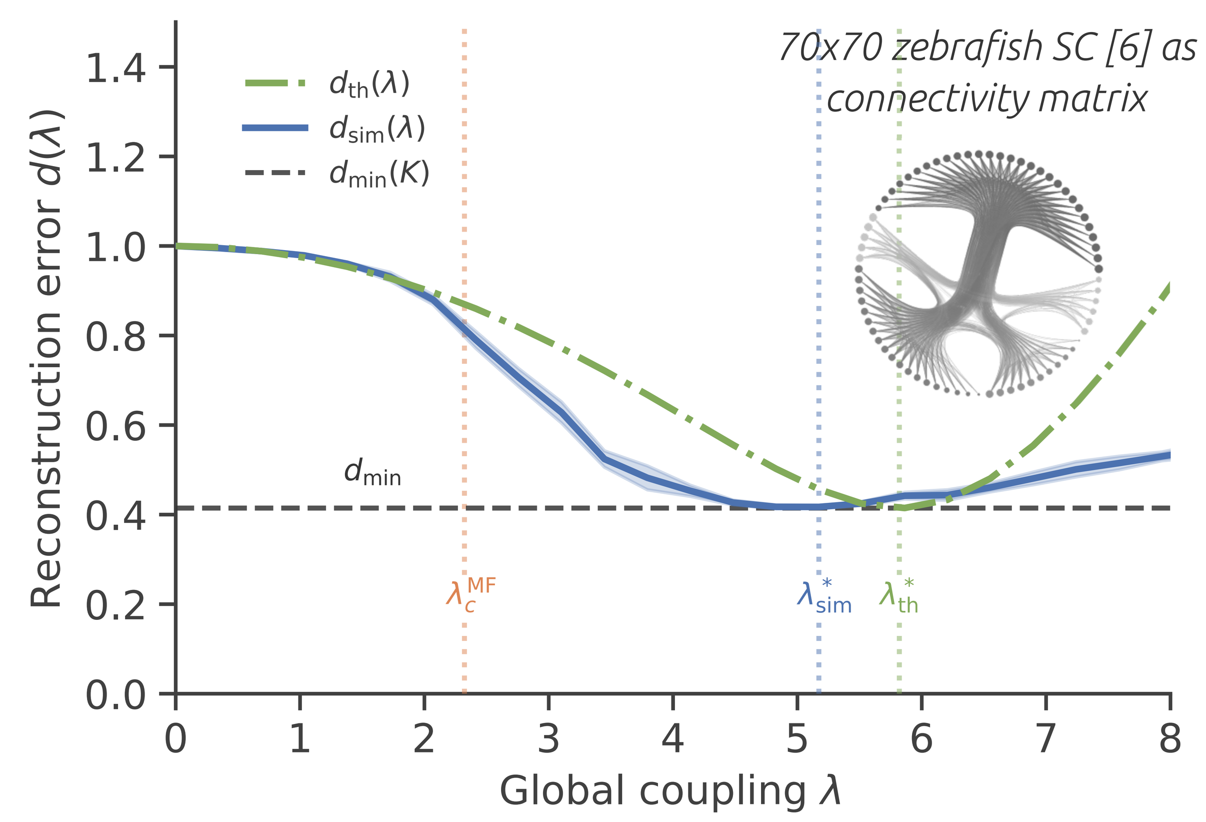
$$\lambda^* = \frac{2\gamma N}{\sqrt{5}}\sqrt{\frac{\mathrm{Tr}[K^2K^\top]}{\mathrm{Tr}[(KK^\top)^2] - \|\mathrm{diag}(KK^\top)\|_F^2}}$$

ii) Lower bound on SC-FC reconstruction error, solely dependent on connectivity matrix

$$d_{\min}(K) = \sqrt{1 - \frac{\mathrm{Tr}^2[K^2K^\top]}{\|K\|_F^2\,\mathrm{Tr}[(KK^\top - \mathrm{diag}(KK^\top))^2]}}$$

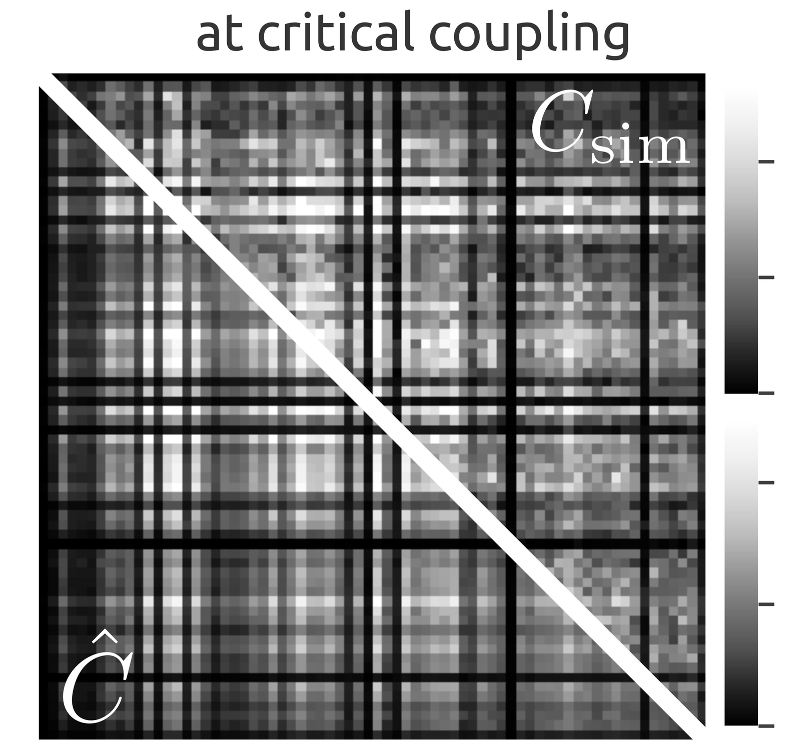
Numerical validations

(a) SC-FC reconstruction error vs global coupling



- 12.6 % of deviation from empirical optimal coupling
- Numerically exact lower bound on reconstruction error

(b) Predicted vs simulated FC at critical coupling



$$r(\hat{C}, C_{\mathrm{sim}})\big|_{\lambda_c^{\mathrm{MF}}} = 0.97 \pm 0.01$$

- 97 % of cosine similarity for FC-FC at the edge of criticality

Takeaways and future work

- We developed an analytical approach for mapping SC to FC, giving:
  - — a closed-form prediction of stationary correlations, valid in the weak-coupling;
  - — an optimal coupling of strongest SC-FC relationship, associated with a lower bound on SC-FC reconstruction error.
- Key biophysical insights:
  - — the dispersion of frequencies lowers imprint of correlations;
  - — direct structural connections are absent from FC;
  - — common-neighbor structure determines FC.
- Building upon...
  - — Is $d_{\min}$ a fundamental lower bound on reconstruction error?

Python Library

Structure Influence on Models of Brain Activity

SIMBA

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[4] B. Prasse et al., "Predicting network dynamics without requiring the knowledge of the interaction graph," Proc. Natl. Acad. Sci. U.S.A., 2022.
[5] C. Murphy et al., "Duality between predictability and reconstructability in complex systems," Nat. Commun., 2024.
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