Realistic clustering patterns in directed geometric networks

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Clustering is one of the most fundamental property of real complex networks. Yet, contrary to other fundamental properties like the degree distribution and degree correlations, very few models are capable of reproducing the patterns of clustering observed in real complex networks (i.e. to go beyond matching the value of the clustering coefficient). Indeed, because the presence of triangles imply three-node interactions, clustering is notoriously difficult to model without recurring to approximations such as an underlying tree-like organization, or to numerical simulations. A successful alternative is to devise network models in latent geometric spaces [1].

By leveraging the triangle inequality of an underlying metric space, the framework of network geometry can reproduce the intricate patterns of clustering observed in real complex networks from pairwise interactions only, as well as several other topological properties [2]. However, this framework has been limited to undirected networks: any straightforward extension to directed networks would require an asymmetric distance function to account for direction, which goes against the definition of any metric space.

We present an elegant general solution to this apparent contradiction. Our model is able to reproduce both the joint distribution of in-degrees and out-degrees and the number of triangles, and includes an additional parameter that tunes the level of reciprocity—the propensity for two directed links to exists between a same pair of nodes—, a fundamental property of real directed networks [3]. It is also amenable to several analytical and semi-analytical calculations. Besides, our methodology can also be applied to non-geometric models with pairwise interactions to control for the level of reciprocity.

Most importantly, we use our approach to show that the even more complex patterns of clustering in directed networks[1] are in fact a byproduct of the joint distribution of in-degree and out-degree, of reciprocity and of the underlying metric space (see Fig. 1). Our contribution offers a rigorous path to extend network geometry to directed networks, thus allowing this powerful approach to be used to study the topology of several real complex systems like the brain, food webs, information networks, and human interactions.

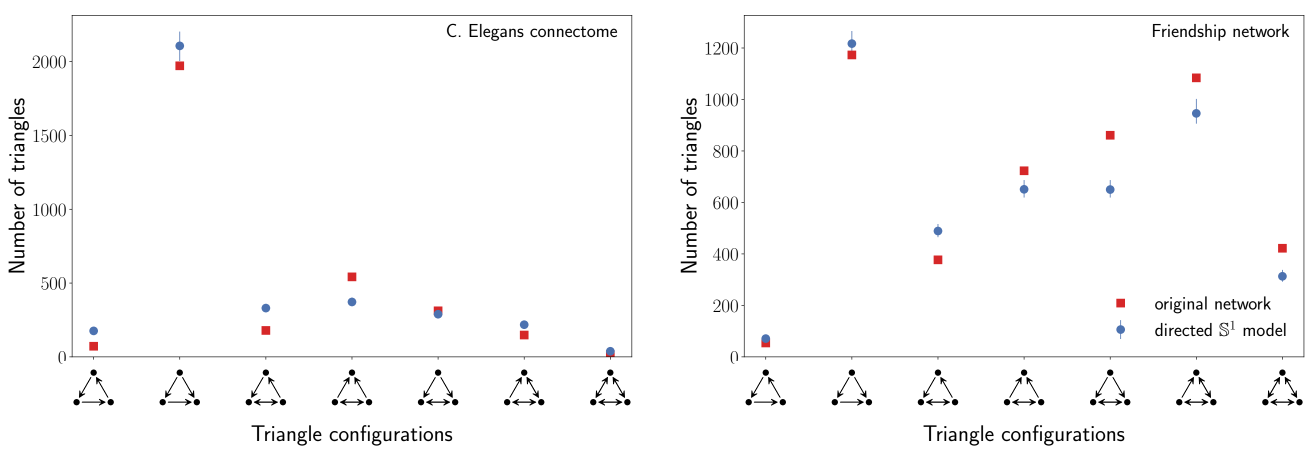

FIG. 1. Triangle spectra observed in the original networks (red squares) with the ones measured in synthetic networks generated by the directed $\mathbb{S}^1$ model (blue circles and error bars indicating quartiles).

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[1] There are 7 possible triangle configurations in directed networks instead of the unique one in undirected networks. Clustering must therefore be characterized minimally by a triangle spectrum as the one shown in Fig. 1 [4].