A threefold approach for reducing the dimension of dynamics on networks: An application to synchronization

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Several complex systems can be modeled as large networks in which the state of the nodes continuously evolves through interactions among neighboring nodes, forming a high-dimensional nonlinear dynamical system. One of the main challenges of Network Science consists in predicting the impact of network topology and dynamical parameters on the evolution of the states and, especially, on the emergence of collective phenomena, such as synchronization [1]. We address this problem by proposing a method that maps high-dimensional (complete) dynamics unto low-dimensional (reduced) dynamics while preserving the most salient features, both topological and dynamical, of the original system. *Our method generalizes recent approaches* [2,3,4] for dimension-reduction by allowing the treatment of complex-valued dynamical variables, heterogeneities in the dynamical parameters as well as modular networks with strongly interacting communities. Most importantly, *we identify three major reduction procedures* whose relative accuracy depends on whether the evolution of the states is mainly determined by the dynamical parameters, the degree sequence, or the adjacency matrix. We use phase synchronization of oscillator networks as a benchmark for our method. As shown in FIG. 1, we successfully *predict the synchronization curves for three phase dynamics* (Winfree, Kuramoto, theta) on the random bipartite model and the stochastic block model. We finally use the threefold dimension-reduction to get analytical insights on chimera states and explosive synchronization for the Kuramoto-Sakaguchi dynamics.

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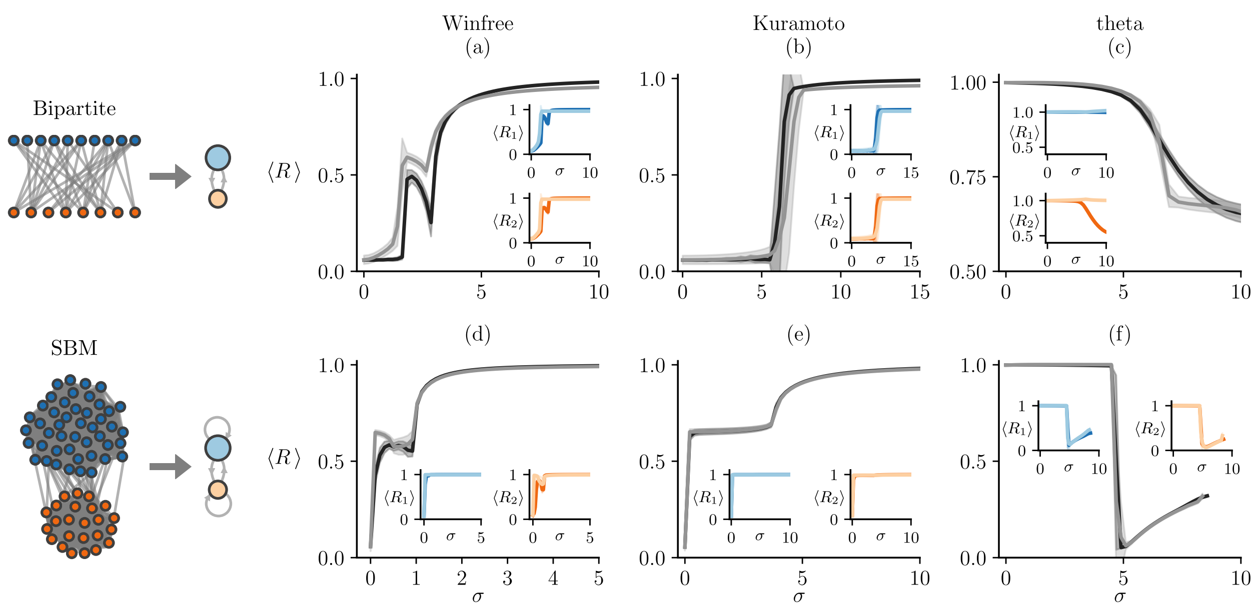

FIG. 1. Comparison of the synchronization curves (average synchronization measure $\langle R \rangle$ vs. the coupling constant σ) between the complete phase dynamics (black lines) and their reduced dynamics (gray lines) on random modular graphs (bipartite and SBM). The complete and reduced dynamics have respective dimension $N = 250$ and $n = 2$. (Insets) Average mesoscopic synchronization measure $\langle R_\mu \rangle$ vs. the coupling constant σ where $\mu \in \{1, 2\}$ is the index of the community. (a, b, d, e) The natural frequencies of the first community $\boldsymbol{\omega}_1$ are drawn from a normal distribution of mean 0.3 and of variance 0.01 and the natural frequencies of the second community $\boldsymbol{\omega}_2$ are chosen to ensure that $\sum_{j=1}^{N} \omega_j = 0$. (c, f) The currents of the first community are drawn from a normal distribution of mean -1.1 and of variance 0.01 and the currents of the second community are drawn from a normal distribution of mean -0.9 and of variance 0.01.

Parameters: $N_1 = 150$ and $N_2 = 100$ are the number of nodes in each communities. For the bipartite, $p_{11} = p_{22} = 0$ and $p_{12} = p_{21} = 0.2$ where $p_{\mu\nu}$ is the probability of a node in community μ to have a connection with a node in community ν. For the SBM, $p_{11} = 0.7$, $p_{22} = 0.5$, and $p_{12} = p_{21} = 0.2$. The synchronization measures are averaged over the second half of the time series, over 50 graphs of the ensembles, over 50 sets of parameters, and over 50 initial conditions randomly chosen from a normal distribution. The shaded region around each line is the standard deviation of the time-averaged synchronization measure.